Numerical Models for the Expanding Universe

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Abstract

This document gives an overview of the equations describing the expansion of a homogeneous and isotropic universe. Common mathematical software, which can be run on a standard computer, is used to numerically solve the equations. The results for different compositions of the universe are presented, identifying the model which best matches recent cosmological observations.

Contents

1 High school mathematics review

1.1 Surface areas and volumes

The surface area A of a sphere with radius r is given by:

$$A = 4\pi r^2$$

The volume V of a sphere with radius r is given by:

$$V = \frac{4}{3}\pi r^3$$

1.2 Derivative of a function

If x is a function of variable t, the derivative of x with respect to t is defined as:

$$\dot{x} = \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

The following calculation rules apply for operations with functions $x(t)$ and $y(t)$:

$$\frac{d}{dt}(xy) = y\frac{dx}{dt} + x\frac{dy}{dt} \tag{1}$$

$$\frac{d}{dt}\left(\frac{x}{y}\right) = \frac{y\frac{dx}{dt} - x\frac{dy}{dt}}{y^2}$$

$$\frac{d}{dt}(x^n) = nx^{n-1}\frac{dx}{dt} \tag{2}$$

1.3 Pythagoras' theorem

If v is the magnitude of a vector $\vec{v}$ with vectorial components (v_x, v_y, v_z) along 3 perpendicular axis, then Pythagoras' theorem states:

$$v^2 = v_x^2 + v_y^2 + v_z^2 \tag{3}$$

2 Evidence of an expanding universe

2.1 Doppler shift

The Doppler effect is a well-known phenomenon for sound waves but equally affects electromagnetic waves as shown in figure 1. When a source moves towards an observer, the emitted waves are compressed and their wavelength gets shorter. A shorter wavelength is equivalent with a higher frequency which explains the high pitch of an approaching ambulance's siren or an approaching train's whistle. Once past the observer, the emitted waves are stretched out and their wavelength consequently get longer. The equivalent lower frequency makes the pitch of the ambulance's siren or the train's whistle drop when the ambulance or train are passing by.

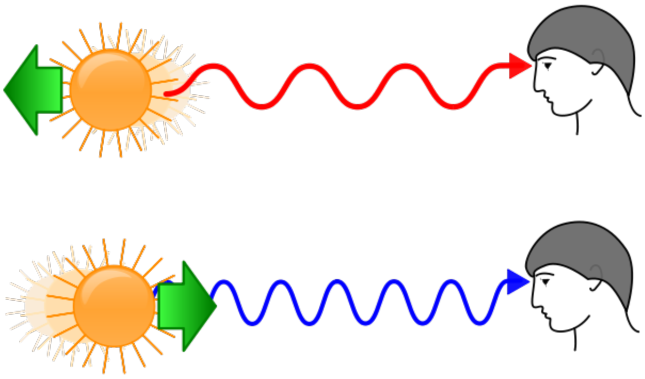

Figure 1: Doppler effect causing a red shift or blue shift in the light of distant galaxies.

The light of stars and galaxies contains spectral lines which are characteristic for their chemical compounds. Compared to the spectral lines of e.g. our sun, those of distant objects are observed at longer or shorter wavelengths due to the Doppler shift. When they are shifted to the longer wavelength side of the spectrum, the light is called red shifted. In the opposite case, it is said to be blue shifted. In 1912, Vesto Slipher [1] was the first astronomer who observed the Doppler shift in the light of what were called faint nebulae at that era. He discovered that, apart from a few exceptions, they all exhibit a redshift and are thus receding away from us.

2.2 Hubble's law

At the beginning of the 19th century, the vastness of the universe was not yet known and space was thought to be not much bigger than the Milky Way. When Edwin Hubble [2] started measuring the distance of faint nebulae in 1922, his observations showed that they are located so far away that they cannot be part of the Milky Way. Only then, astronomers realized that the faint nebulae are galaxies themselves located far away from us.

In 1929, Hubble combined his work with the work of Vesto Slipher and discovered a roughly linear relation between radial velocity (redshift) and distance, known as Hubble's law. Figure 2 is a copy of the figure which appeared in Hubble's original paper. The proportionality factor between speed $v(t)$ and distance $r(t)$ is called the Hubble parameter $H(t)$:

$$\boxed{v = Hr} \tag{4}$$

The time dependency of the Hubble parameter $H(t)$ expresses that it is not necessarily constant over time. The term "Hubble constant" therefore only refers to the present value of the Hubble parameter $H(t_0)$.

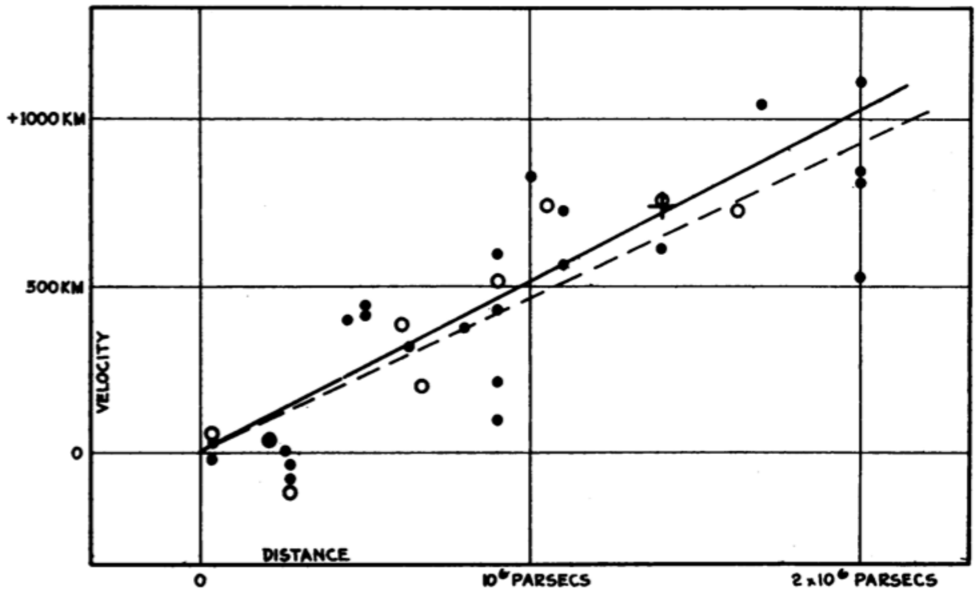

Figure 2: Velocity-distance relation among extra-galactic nebulae as it appeared in Hubble's original paper dated 1929.

[1] Vesto Slipher (1875 - 1969) was an American astronomer who worked his entire professional career at Lowell Observatory in Flagstaff, Arizona.

[2] Edwin Hubble (1889 - 1953) was an American astronomer who worked the majority of his professional career at Mount Wilson Observatory in Pasadena, California.

2.3 Comoving coordinate systems

Assume that $\vec{x}$, $\vec{y}$ and $\vec{z}$ are the unit vectors along the 3 axes of a cartesian coordinate system. As the universe expands, the comoving coordinate system expands together with space and after a certain time, the unit vectors have become $\vec{x'}$, $\vec{y'}$ and $\vec{z'}$. In a homogeneous and isotropic universe, both sets of unit vectors are related to each other by a single time dependent scale factor $a(t)$:

$$\vec{x'} = a(t)\vec{x}$$

$$\vec{y'} = a(t)\vec{y}$$

$$\vec{z'} = a(t)\vec{z}$$

A homogeneous universe is a universe which looks the same at every location. An isotropic universe is a universe which looks the same in every direction. Obviously, the solar system does not have the same appearance when observed from Earth compared to e.g. from Jupiter. And when admiring the sky on a dark cloudless night, it shows differently in northern direction compared to southern direction. The universe can therefore only be considered homogeneous and isotropic on sufficiently large [3] scales.

3 The Friedmann equation

3.1 In terms of the scale factor

Based on Einstein's theory of general relativity, the Russian cosmologist Alexander Friedmann [4] derived the equation which describes the evolution of the scale factor $a(t)$ over time for a homogeneous and isotropic universe.

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon - \frac{kc^2}{a^2} + \frac{\Lambda}{3}} \tag{5}$$

The Friedmann equation is a first order differential equation in which the scale factor $a(t)$ and the energy density $\epsilon(t)$ are time dependent variables. All the other parts are constants with G being the gravitational constant and c the speed of light.

The factor k is a dimensionless number and describes the curvature of space. Depending on whether it is negative, zero or positive, space has a hyperbolic, flat or spherical geometry as summarized in table 1.

Curvature	Geometry	Triangle	Circle	Universe
$k < 0$	hyperbolic	$\Sigma\alpha < \pi$	$c > 2\pi r$	open
$k = 0$	flat	$\Sigma\alpha = \pi$	$c = 2\pi r$	flat
$k > 0$	spherical	$\Sigma\alpha > \pi$	$c < 2\pi r$	closed

Table 1: Classification of geometries depending the value of curvature k.

[3] Cosmologists typically use a threshold of 100 Mpc as the scale beyond which the universe can be considered homogeneous and isotropic.

[4] Alexander Friedmann (1888 - 1925) was a Russian physicist and mathematician who lived and worked in Saint Petersburg.

While 3-dimensional hyperbolic or spherical geometries are not easy to visualize, their 2-dimensional analogies as depicted in figure 3 help to understand the essential differences.

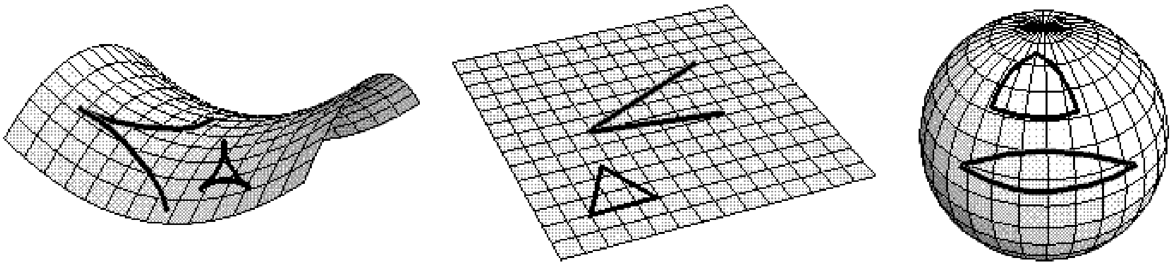

Figure 3: 2D analogies of hyperbolic, flat and spherical geometries.

The term Λ in Friedmann's equation is the cosmological constant which Einstein added to his field equations of general relativity in 1917. The reason was that, after having published his theory in 1915, Einstein applied it to the universe and soon realized all solutions were describing a universe which is expanding or contracting at any given moment in time. Such a dynamic universe could not be reconciled with widespread belief at that time that the universe was static. The introduction of a cosmological constant made it possible to achieve a static solution.

When Vesto Slipher and Edwin Hubble discovered that the universe is not static but expanding, Einstein dropped the cosmological constant again and reverted to his original field equations. This remained unchanged until 1998 when new observational evidence showed that the expansion of the universe is accelerating, something that cannot be explained without cosmological constant. Although its exact nature is still unknown, whatever the cosmological constant represents is now widely accepted as an important component of our universe.

3.2 In terms of the Hubble parameter

Introducing the scale factor $a(t)$ in Hubble's law (4) makes it possible to write the Hubble parameter $H(t)$ in terms of the scale factor and its first order derivative:

$$v = Hr$$

$$\frac{dr}{dt} = Hr$$

$$\frac{d}{dt}(ar_0) = Har_0$$

$$\frac{da}{dt}r_0 = Har_0$$

$$\boxed{H = \frac{\dot{a}}{a}} \tag{6}$$

Friedmann equation (5) can then be rewritten as:

$$\boxed{H^2 = \frac{8\pi G}{3c^2}\epsilon - \frac{kc^2}{a^2} + \frac{\Lambda}{3}} \tag{7}$$

3.3 In terms of the density parameter

The critical density $\epsilon_c(t)$ is defined as:

$$\boxed{\epsilon_c = \frac{3c^2}{8\pi G}H^2} \tag{8}$$

The density parameter $\Omega(t)$ is defined as:

$$\boxed{\Omega = \frac{\epsilon}{\epsilon_c}} \tag{9}$$

Combining equations (8) and (9) gives:

$$\boxed{\epsilon = \frac{3c^2H^2}{8\pi G}\Omega} \tag{10}$$

It is possible to define a constant energy density ϵ_Λ related to the cosmological constant as:

$$\boxed{\epsilon_\Lambda = \frac{c^2}{8\pi G}\Lambda} \tag{11}$$

Combining equation (10) applied to ϵ_Λ and equation (11) yields:

$$\frac{c^2}{8\pi G}\Lambda = \frac{3c^2H^2}{8\pi G}\Omega_\Lambda$$

$$\frac{\Lambda}{3} = H^2\Omega_\Lambda$$

Substituting equation (10) and above expression for $\frac{\Lambda}{3}$ in Friedmann equation (7) leads to:

$$H^2 = \frac{8\pi G}{3c^2}\epsilon - \frac{kc^2}{a^2} + \frac{\Lambda}{3}$$

$$H^2 = \frac{8\pi G}{3c^2}\frac{3c^2H^2}{8\pi G}\Omega - \frac{kc^2}{a^2} + H^2\Omega_\Lambda$$

$$H^2 = H^2\Omega + H^2\Omega_\Lambda - \frac{kc^2}{a^2}$$

$$(1 - \Omega - \Omega_\Lambda)\,H^2 = -\frac{kc^2}{a^2}$$

$$\boxed{1 - \Omega - \Omega_\Lambda = -\frac{kc^2}{a^2H^2}} \tag{12}$$

For an open universe ($k < 0$), equation (12) results in:

$$1 - \Omega - \Omega_\Lambda > 0$$

$$\Omega + \Omega_\Lambda < 1$$

For a flat universe ($k = 0$), equation (12) results in:

$$1 - \Omega - \Omega_\Lambda = 0$$

$$\Omega + \Omega_\Lambda = 1$$

For a closed universe ($k > 0$), equation (12) results in:

$$1 - \Omega - \Omega_\Lambda < 0$$

$$\Omega + \Omega_\Lambda > 1$$

The constraints imposed on the total density parameter by the curvature of space are summarized in table 2.

Curvature	Geometry	Universe	Density
$k < 0$	hyperbolic	open	$\Omega + \Omega_\Lambda < 1$
$k = 0$	flat	flat	$\Omega + \Omega_\Lambda = 1$
$k > 0$	spherical	closed	$\Omega + \Omega_\Lambda > 1$

Table 2: Constraints on the total density parameter for the different geometries.

4 The fluid equation

The volume $V(t)$ of a sphere with radius $r(t) = a(t)r_0$ is:

$$V(t) = \frac{4}{3}\pi a(t)^3 r_0^3$$

Its time derivative is consequently:

$$\frac{dV}{dt} = \frac{4}{3}\pi r_0^3 3a^2 \frac{da}{dt}$$

$$\frac{dV}{dt} = 3V\frac{\dot{a}}{a} \tag{13}$$

The internal energy $E(t)$ of a sphere with volume $V(t)$ and energy density $\epsilon(t)$ is:

$$E(t) = V(t)\epsilon(t)$$

Its time derivative is consequently:

$$\frac{dE}{dt} = V\frac{d\epsilon}{dt} + \epsilon\frac{dV}{dt}$$

$$\frac{dE}{dt} = V\dot{\epsilon} + \epsilon 3V\frac{\dot{a}}{a}$$

$$\frac{dE}{dt} = V\left(\dot{\epsilon} + 3\frac{\dot{a}}{a}\epsilon\right) \tag{14}$$

The first law of thermodynamics states that the change in internal energy dE of a system expanding in a quasi-static process is the sum of the amount of heat dQ supplied to the system and the work done on the system $-PdV$:

$$dE = dQ - PdV$$

$$dQ = dE + PdV \tag{15}$$

In a homogeneous universe there is no heat transfer in or out of a co-moving volume (adiabatic expansion). Combining equations (13), (14) and (15) then yields:

$$dQ = 0$$

$$dE + PdV = 0$$

$$\frac{dE}{dt} + P\frac{dV}{dt} = 0$$

$$V\left(\dot{\epsilon} + 3\frac{\dot{a}}{a}\epsilon\right) + P3V\frac{\dot{a}}{a} = 0$$

$$\boxed{\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0} \tag{16}$$

Equation (16) can also be rewritten as:

$$\dot{\epsilon}\frac{a}{\dot{a}} = -3(\epsilon + P)$$

5 The acceleration equation

The acceleration equation is not a new equation but rather a combination of the Friedmann equation and the fluid equation. Taking the time derivative of Friedmann equation (5), dividing by $2a\dot{a}$ and substituting the fluid equation gives:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon - \frac{kc^2}{a^2} + \frac{\Lambda}{3}$$

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\epsilon a^2 - kc^2 + \frac{\Lambda a^2}{3}$$

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3c^2}\left(\dot{\epsilon}a^2 + 2\epsilon a\dot{a}\right) + \frac{2a\dot{a}\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}\left(\dot{\epsilon}\frac{a}{\dot{a}} + 2\epsilon\right) + \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}\left(-3\epsilon - 3P + 2\epsilon\right) + \frac{\Lambda}{3}$$

$$\boxed{\frac{\ddot{a}}{a} = \frac{-4\pi G}{3c^2}\left(\epsilon + 3P\right) + \frac{\Lambda}{3}} \tag{17}$$

If there would be no cosmological constant Λ in equation (17), its outcome would never be positive, taking into account that the energy density $\epsilon(t)$ and pressure $P(t)$ of matter and radiation are non-negative quantities. This would imply that the universe expands at a constant or decreasing rate, but both the Supernova Cosmology Project [5] and the High-Z Supernova Search Team [6] independently discovered in 1998 that the expansion of the universe is actually accelerating. The cosmological constant is the only possibility to achieve a positive value for equation (17) and explain the observational evidence for an accelerating expansion.

6 The equation of state

6.1 General form

For the kind of substances which are playing a role in cosmological calculations, the equation of state is a linear relationship between pressure $P(t)$ and energy density $\epsilon(t)$:

$$\boxed{P = w\epsilon} \tag{18}$$

Fluid equation (16) can then be rewritten as:

[5] http://supernova.lbl.gov

[6] https://www.cfa.harvard.edu/supernova/home.html

$$\dot{\epsilon} + 3\frac{\dot{a}}{a}\left(\epsilon + w\epsilon\right) = 0$$

$$\dot{\epsilon} + 3\left(1 + w\right)\frac{\dot{a}}{a}\epsilon = 0$$

Using calculation rules for derivatives (1) and (2), it can be shown that this is equivalent with:

$$\frac{1}{a^{3(1+w)}}\frac{d}{dt}\left(\epsilon a^{3(1+w)}\right) = 0$$

$$\frac{d}{dt}\left(\epsilon a^{3(1+w)}\right) = 0$$

$$\epsilon a^{3(1+w)} = cst$$

$$\epsilon \propto \frac{1}{a^{3(1+w)}} \tag{19}$$

6.2 Pressure of a particle gas

The momentum $\vec{p}$ of a particle with energy E moving at a velocity $\vec{v}$ can be defined as:

$$\vec{p} = \frac{E}{c^2}\vec{v}$$

This definition is valid both for massive particles moving at non-relativistic speeds as for massless particles moving at speeds close the the speed of light.

Decomposing the vectors in 3 perpendicular components yields in x-direction:

$$p_x = \frac{E}{c^2}v_x$$

Similar equations can be written down for p_y and p_z.

Consider a cubical container with side L and volume V, filled with a gas consisting of N particles randomly moving around at an average speed $\bar{v}$. When a particle elastically collides with a wall of the container perpendicular to the x-direction, its momentum before and after the collision will be the same but with opposite sign. Its change in momentum Δp is therefore:

$$\Delta p = p_x - (-p_x) = 2p_x$$

Taking the definition of momentum into account, this becomes:

$$\Delta p = 2\frac{E}{c^2}v_x$$

On average, the particle will collide with the wall of the container at intervals Δt given by:

$$\Delta t = \frac{2L}{v_x}$$

The force exerted on the wall of the container by that particle is:

$$F_i = \frac{\Delta p}{\Delta t} = \frac{E}{L}\frac{v_x^2}{c^2}$$

Summarizing over all the particles in the container, the total force exerted is:

$$F = \sum_{i=1}^{N} F_i = \frac{NE}{L}\frac{\bar{v}_x^2}{c^2}$$

As the particles are randomly moving around, there is no preferred direction and statistically:

$$\bar{v}_x^2 = \bar{v}_y^2 = \bar{v}_z^2$$

Substituting this equality in Pythagoras' theorem (3) yields:

$$\bar{v}_x^2 = \frac{1}{3}\bar{v}^2$$

The pressure exerted by the particles in the container is the force per unit of area or:

$$P = \frac{F}{L^2} = \frac{1}{3}\frac{NE}{L^3}\frac{\bar{v}^2}{c^2}$$

In terms of the energy density ϵ this becomes:

$$P = \frac{1}{3}\frac{\bar{v}^2}{c^2}\epsilon$$

By comparison with the general form of the equation of state (18), it can be concluded that for a gas of particles:

$$\boxed{w = \frac{1}{3}\frac{\bar{v}^2}{c^2}} \tag{20}$$

6.3 Non-relativistic matter

In cosmology, non-relativistic matter is the term used for massive particles moving at speeds considerably less than the speed of light. On earth, all matter is made of protons and neutrons and called baryonic [7] matter. Several research groups have shown that the estimated amount of baryonic matter in the universe is not always in agreement with the expected amount of matter based on particular observations. For example, the constant rotational velocity of galaxies at distances far away from their centre cannot be explained by their baryonic content alone. Gravitational lensing is another phenomenon which indicates there is vastly more matter in the universe than we are able to see. This unknown non-baryonic matter is commonly called dark matter, given that it cannot directly be observed with today's means. Recent estimates yield a baryonic matter contribution to $\Omega_{m,0}$ of 4.8% and a dark matter contribution of 26.2%.

For non-relativistic matter, $\bar{v}^2 << c^2$ and consequently, based on equations (20) and (18):

$$\boxed{w_m \approx 0 \Leftrightarrow P_m \approx 0}$$

Equation (19) can then be written as:

$$\epsilon_m \propto \frac{1}{a^3}$$

Denoting the present value of the scale factor as a_0 and the present value of the matter energy density as $\epsilon_{m,0}$, this proportionality can also be expressed as:

$$\epsilon_m = \left(\frac{a_0}{a}\right)^3 \epsilon_{m,0} \tag{21}$$

6.4 Radiation

In cosmology, radiation is the term used for massless particles moving at speeds close to the speed of light. Photons are the obvious example of such particles but in a cosmological context, neutrinos are also counted as a form of radiation. Recent research has shown that neutrinos might have a rest mass but still sufficiently small to consider them as massless particles without invalidating the theory.

For radiation, $\bar{v} = c$ and consequently, based on equations (20) and (18):

$$\boxed{w_r = \frac{1}{3} \Leftrightarrow P_r = \frac{1}{3}\epsilon_r}$$

Equation (19) can then be written as:

$$\epsilon_r \propto \frac{1}{a^4}$$

Denoting the present value of the scale factor as a_0 and the present value of the radiation energy density as $\epsilon_{r,0}$, this proportionality can also be expressed as:

[7] Formally, baryons are particles formed by 3 quarks unlike mesons which consist of 2 quarks.

$$\epsilon_r = \left(\frac{a_0}{a}\right)^4 \epsilon_{r,0} \tag{22}$$

6.5 The cosmological constant

Applying fluid equation (16) to the cosmological constant gives:

$$\dot{\epsilon_\Lambda} + 3\frac{\dot{a}}{a}\left(\epsilon_\Lambda + P_\Lambda\right) = 0$$

As ϵ_Λ is a constant by definition, its time derivative $\dot{\epsilon_\Lambda}$ is zero and consequently:

$$3\frac{\dot{a}}{a}\left(\epsilon_\Lambda + P_\Lambda\right) = 0$$

$$\epsilon_\Lambda + P_\Lambda = 0$$

$$\boxed{P_\Lambda = -\epsilon_\Lambda}$$

This shows that for whatever the cosmological constant represents:

$$\boxed{w_\Lambda = -1}$$

7 Age of the universe

7.1 Calculation method

Applying equation (10) at the present time for respectively non-relativistic matter, radiation and the cosmological constant gives:

$$\epsilon_{m,0} = \frac{3c^2H_0^2}{8\pi G}\Omega_{m,0} \tag{23}$$

$$\epsilon_{r,0} = \frac{3c^2H_0^2}{8\pi G}\Omega_{r,0} \tag{24}$$

$$\epsilon_{\Lambda,0} = \frac{3c^2H_0^2}{8\pi G}\Omega_{\Lambda,0} \tag{25}$$

Substituting equation (23) in equation (21), the non-relativistic matter energy density $\epsilon_m(t)$ can be written as:

$$\epsilon_m = \left(\frac{a_0}{a}\right)^3 \frac{3c^2H_0^2}{8\pi G}\Omega_{m,0} \tag{26}$$

Substituting equation (24) in equation (22), the radiation energy density $\epsilon_r(t)$ can be written as:

$$\epsilon_r = \left(\frac{a_0}{a}\right)^4 \frac{3c^2 H_0^2}{8\pi G}\Omega_{r,0} \tag{27}$$

It is possible to define the present curvature density parameter $\Omega_{k,0}$ as:

$$\boxed{\Omega_{k,0} = -\frac{kc^2}{a_0^2 H_0^2}}$$

The curvature term of Friedmann equation (5) can then be written as:

$$-\frac{kc^2}{a^2} = H_0^2 \left(\frac{a_0}{a}\right)^2 \Omega_{k,0} \tag{28}$$

The cosmological constant energy density ϵ_Λ does not change over time the way it was defined in equation (11). Its value at the present time equals its value at any other time, otherwise Λ would not be a constant. Combining equations (11) and (25), the cosmological constant term of Friedmann equation (5) can be written as:

$$\epsilon_\Lambda = \epsilon_{\Lambda,0}$$

$$\frac{c^2}{8\pi G}\Lambda = \frac{3c^2 H_0^2}{8\pi G}\Omega_{\Lambda,0}$$

$$\frac{\Lambda}{3} = H_0^2 \Omega_{\Lambda,0} \tag{29}$$

Substituting equations (26), (27), (28) and (29) in Friedmann equation (5) yields:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}(\epsilon_r + \epsilon_m) - \frac{kc^2}{a^2} + \frac{\Lambda}{3}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\left[\frac{3c^2 H_0^2}{8\pi G}\left(\frac{a_0}{a}\right)^4 \Omega_{r,0} + \frac{3c^2 H_0^2}{8\pi G}\left(\frac{a_0}{a}\right)^3 \Omega_{m,0}\right]$$

$$+H_0^2 \left(\frac{a_0}{a}\right)^2 \Omega_{k,0} + H_0^2 \Omega_{\Lambda,0}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left[\left(\frac{a_0}{a}\right)^4 \Omega_{r,0} + \left(\frac{a_0}{a}\right)^3 \Omega_{m,0} + \left(\frac{a_0}{a}\right)^2 \Omega_{k,0} + \Omega_{\Lambda,0}\right]$$

It is possible to define a relative scale factor $x(t)$ as the ratio between the scale factor $a(t)$ and the scale factor at the present time $a(t_0)$:

$$x = \frac{a}{a_0} \tag{30}$$

The first order time derivative of the relative scale factor then is:

$$\dot{x} = \frac{\dot{a}}{a_0}$$

Changing over to the relative scale factor $x(t)$ yields:

$$\left(\frac{\dot{x}}{x}\right)^2 = H_0^2\left(\frac{\Omega_{r,0}}{x^4} + \frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{k,0}}{x^2} + \Omega_{\Lambda,0}\right) \tag{31}$$

$$\left(\frac{\dot{x}}{x}\right)^2 = \frac{H_0^2}{x^4}\left(\Omega_{r,0} + \Omega_{m,0}x + \Omega_{k,0}x^2 + \Omega_{\Lambda,0}x^4\right)$$

$$\dot{x}^2 = \frac{H_0^2}{x^2}\left(\Omega_{r,0} + \Omega_{m,0}x + \Omega_{k,0}x^2 + \Omega_{\Lambda,0}x^4\right)$$

$$\frac{dx}{dt} = \frac{H_0}{x}\sqrt{\Omega_{r,0} + \Omega_{m,0}x + \Omega_{k,0}x^2 + \Omega_{\Lambda,0}x^4}$$

$$\frac{dt}{dx} = \frac{x}{H_0\sqrt{\Omega_{r,0} + \Omega_{m,0}x + \Omega_{k,0}x^2 + \Omega_{\Lambda,0}x^4}}$$

$$\boxed{t_0 = \int_0^1 \frac{x dx}{H_0\sqrt{\Omega_{r,0} + \Omega_{m,0}x + \Omega_{k,0}x^2 + \Omega_{\Lambda,0}x^4}}} \tag{32}$$

Taking into account that the relative scale factor at the present time $x(t_0)$ equals 1 and that $\dot{x}(t_0)/x(t_0)$ is equal to the Hubble constant H_0, equation (31) applied to the present time t_0 gives:

$$H_0^2 = H_0^2\left(\Omega_{r,0} + \Omega_{m,0} + \Omega_{k,0} + \Omega_{\Lambda,0}\right)$$

$$\boxed{1 = \Omega_{r,0} + \Omega_{m,0} + \Omega_{k,0} + \Omega_{\Lambda,0}}$$

Example code to numerically solve integral (32) with MATLAB [8] is given in the appendices. The function defined in appendix A.1.1 returns the solution for all values of the Hubble constant H_0 supplied as first argument, using the values of present density parameters $\Omega_{m,0}$, $\Omega_{r,0}$ and $\Omega_{\Lambda,0}$ supplied as second to fourth argument. The function is used by the code in appendix A.1.2 to plot the age of the universe t_0 as a function of the Hubble constant H_0 for different composition models. The model parameters are read from a user-selectable file which needs to be structured as shown in appendices A.1.3 and A.1.4.

7.2 Results

7.2.1 Without cosmological constant

Figure 4 plots equation (32) for a nummer of matter dominated universes without cosmological constant. For values of H_0 obtained from recent observations, e.g. via the Planck and WMAP satellites, the topmost curve gives a value for t_0 of about 13 billion years which is still less than the age of the oldest

[8] https://www.mathworks.com

known object in the universe [9]. This shows that models without cosmological constant are not very well matching current observations. The re-introduction of the cosmological constant in Einstein's equations provides a way to bridge the gap between theory and observations.

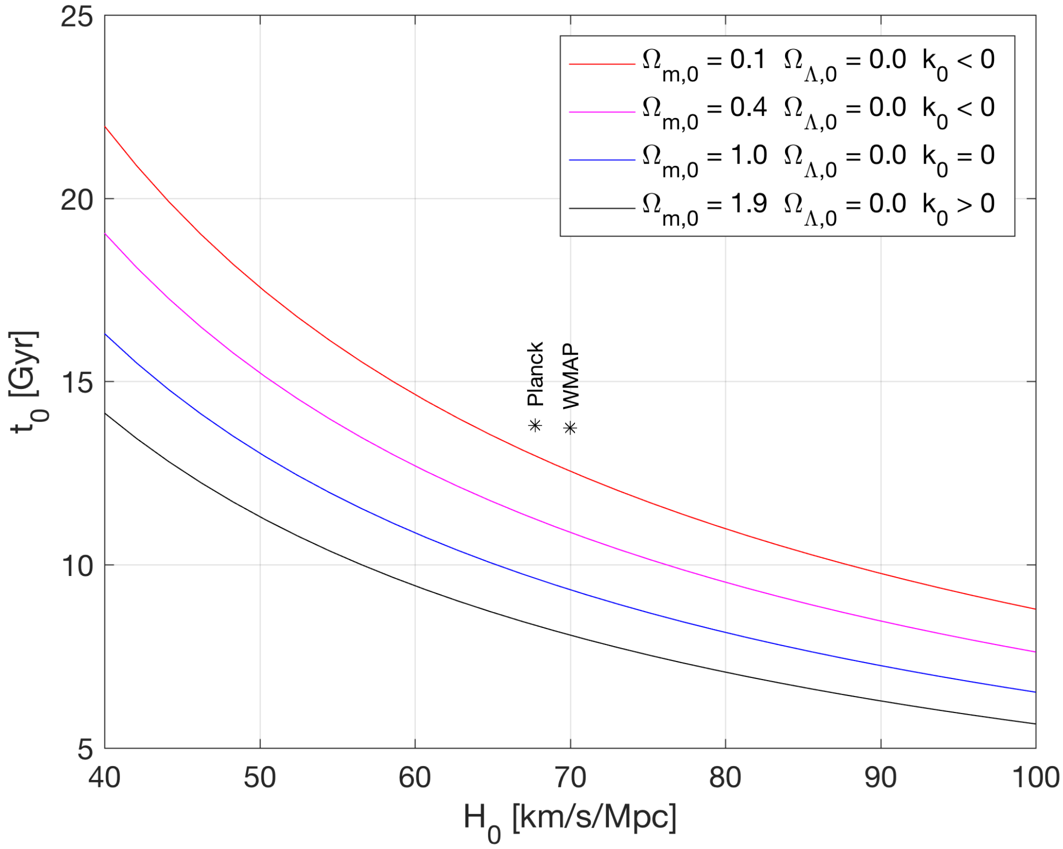

Figure 4: The age of the universe t_0 as a function of the Hubble constant H_0 for different matter dominated ($\Omega_{r,0} = 0$) models without cosmological constant ($\Omega_{\Lambda,0} = 0$).

7.2.2 With flat geometry

Figure 5 plots equation (32) for a nummer of matter dominated universes with flat or Euclidean geometry. Recent observational data from the Planck and WMAP satellites neighbour the magenta curve, representing the model which is widely accepted as best fitting the observations. It is called the ΛCDM model, shorthand for Λ Cold Dark Matter, referring to its main constituents.

[9] The oldest and most distant known galaxy GN-z11 is observed in the constellation Ursa Major as it existed 13.4 billion years ago.

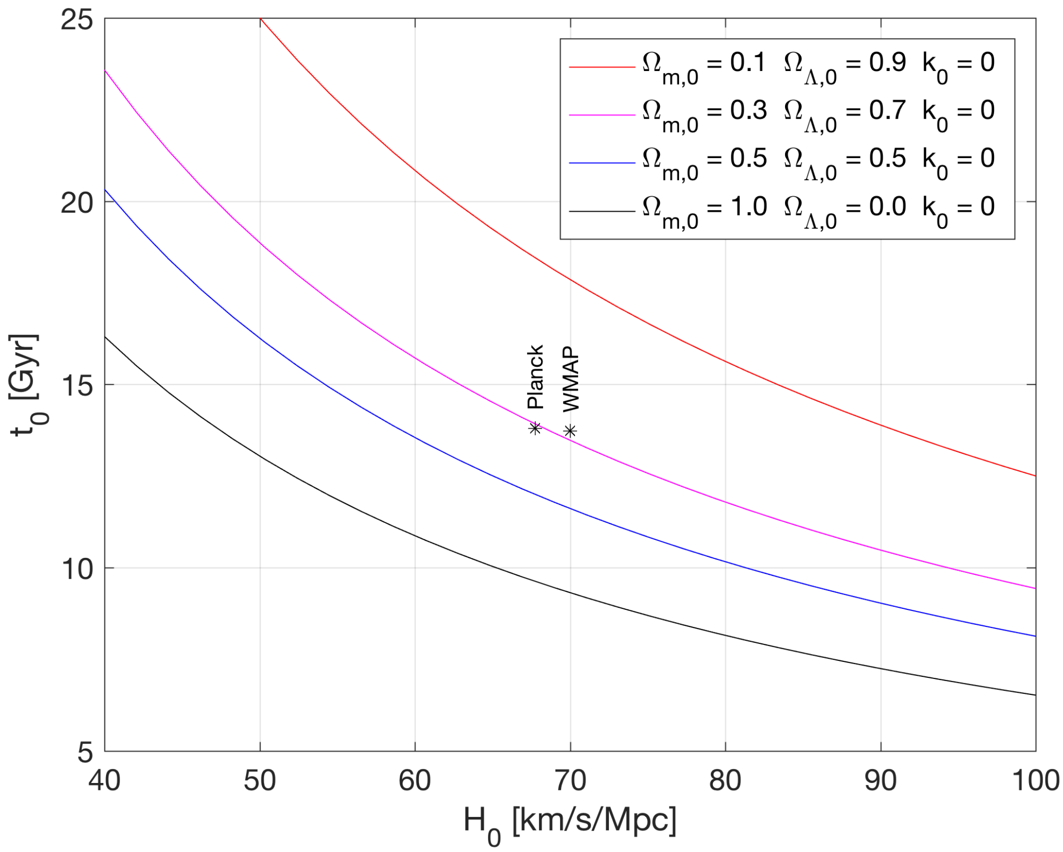

Figure 5: The age of the universe t_0 as a function of the Hubble constant H_0 for different matter dominated ($\Omega_{r,0} = 0$) models with flat geometry ($k_0 = 0$).

7.2.3 The benchmark model

The parameters of the ΛCDM model [10] are summarized in table 3.

Ingredient	Ω_0
photons	$\Omega_{\gamma,0} = 5.35 \times 10^{-5}$
neutrinos	$\Omega_{\nu,0} = 3.65 \times 10^{-5}$
total radiation	$\Omega_{r,0} = 9.00 \times 10^{-5}$
baryonic matter	$\Omega_{b,0} = 0.048$
dark matter	$\Omega_{d,0} = 0.262$
total matter	$\Omega_{m,0} = 0.31$
cosmological constant	$\Omega_{\Lambda,0} \approx 0.69$

Table 3: Parameters of the ΛCDM model.

This leads to a relative distribution of the components of the universe as shown in figure 6. The term dark energy refers to the cosmological constant, giving it a physical meaning while at the same time expressing it is not yet well understood. It is noteworthy that about 95% of the content of the universe is of unknown origin!

[10] Barbara Ryden, *Introduction to Cosmology*, 2nd Edition, page 96

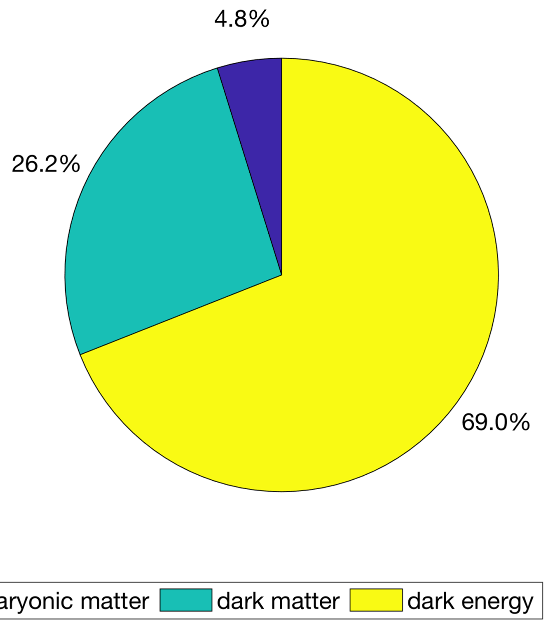

Figure 6: Relative abundance of the different components of the universe according the benchmark model.

8 Expansion of the universe

8.1 Calculation method

Substituting equations (26), (27) and (29) in acceleration equation (17) yields:

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3c^2}\left(\epsilon_r + \epsilon_m + 3P_r + 3P_m\right) + \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3c^2}\left(\epsilon_r + \epsilon_m + \epsilon_r\right) + \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3c^2}\left[2\frac{3c^2 H_0^2}{8\pi G}\left(\frac{a_0}{a}\right)^4 \Omega_{r,0} + \frac{3c^2 H_0^2}{8\pi G}\left(\frac{a_0}{a}\right)^3 \Omega_{m,0}\right] + H_0^2\Omega_{\Lambda,0}$$

Changing over to the relative scale factor $x(t)$ again, as defined in equation (30), yields:

$$\frac{\ddot{x}}{x} = -H_0^2\left(\frac{\Omega_{r,0}}{x^4} + \frac{\Omega_{m,0}}{2x^3} - \Omega_{\Lambda,0}\right)$$

$$\boxed{\ddot{x} = -H_0^2\left(\frac{\Omega_{r,0}}{x^3} + \frac{\Omega_{m,0}}{2x^2} - \Omega_{\Lambda,0}x\right)}$$

This second order differential equation can be solved numerically by converting it into a system of 2 first order differential equations:

$$\begin{cases} \dot{x} & = y \\ \dot{y} & = -H_0^2 \left(\frac{\Omega_{r,0}}{x^3} + \frac{\Omega_{m,0}}{2x^2} - \Omega_{\Lambda,0} x \right) \end{cases}$$

Taking equation (6) and the definition of the relative scale factor (30) into account, following initial conditions apply:

$$\begin{cases} x(t_0) & = 1 \\ \dot{x}(t_0) & = H_0 \end{cases}$$

Example code to numerically solve the equations with MATLAB is given in the appendices. The function defined in appendix A.2.1 implements the system of first order differential equations and is used by the code in appendix A.2.2 to plot the relative scale factor as a function of time for different models of the universe. The model parameters are read from the file in appendix A.2.3.

8.2 Results

The benchmark cosmological model, represented by the red curve in figure 7, combines (cold dark) matter and a cosmological constant with a flat or Euclidean geometry. It is commonly denoted as the ΛCDM model and best fits the observations. According to the benchmark model, the present universe is 13.7 Gyr old. Its expansion accelerates indefinitely and will eventually affect space at small scales and even rip atoms apart. Whether the universe will really end in such a Big Rip cannot be determined as nobody knows how the cosmological constant or whatever it represents behaves on the long run.

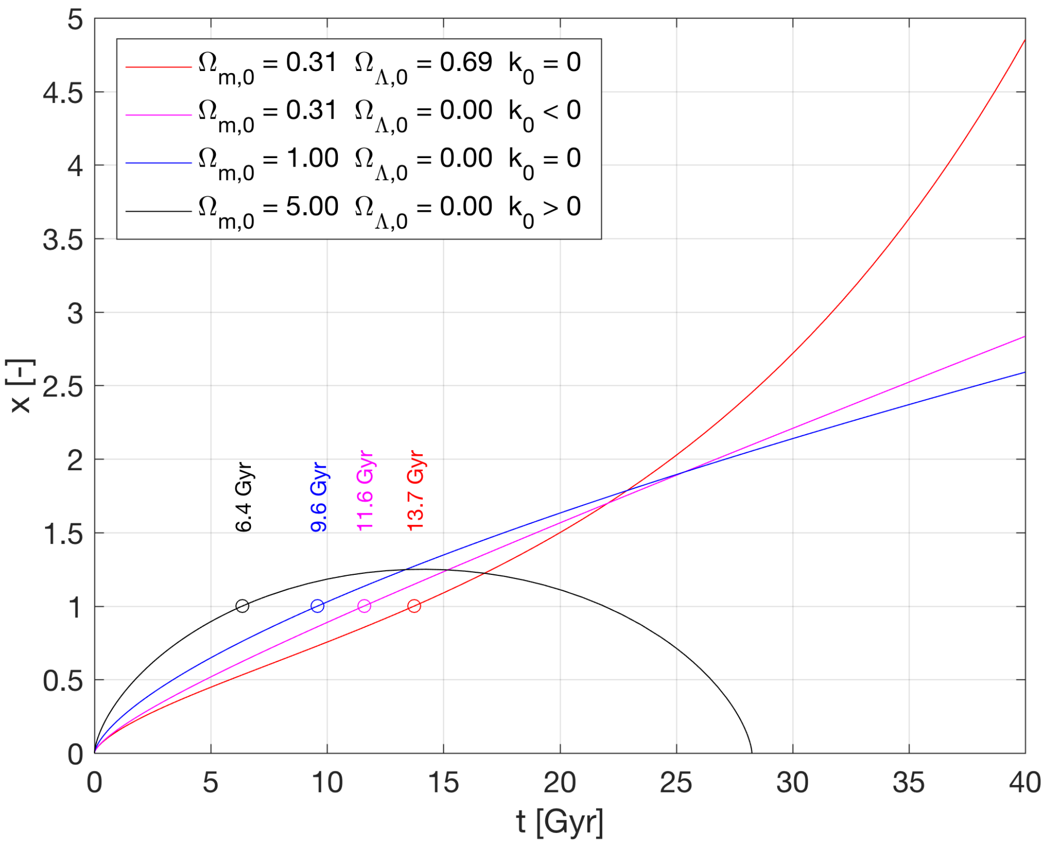

Figure 7: Relative scalefactor as a function of time for different matter dominated ($\Omega_{r,0} = 0$) models.

The other models shown in figure 7 are all mater dominated models without cosmological constant, but with different geometries. The magenta curve represents a negatively curved universe, i.e. one with hyperbolic geometry. This open universe expands eternally at a rate which reaches a constant value at late times. Its ultimate fate is a big cold void, sometimes called the Big Chill, when celestial objects will have died out and moved so far apart that one object's remnant light cannot reach any other object anymore.

The blue curve represents a universe with a density which exactly matches the critical density. In absence of a cosmological constant, it consequently has a flat or Euclidean geometry. The expansion of this flat universe continues eternally at a rate which slows down and approaches zero in the infinite future. The model is known as the Einstein - de Sitter model.

The black curve represents a universe which contains enough matter to allow gravity to stop and reverse its expansion. As a result, it eventually collapses back onto itself in a Big Crunch after about 28 Gyr of existence. This closed universe has positive curvature, i.e. a spherical geometry, and would presently be 6.4 Gyr old and still in its expansion phase.

The long term evolution of the expansion rate can better be understood by plotting it as a function of time like in figure 8, showing that the magenta and blue universes indeed level off to a constant expansion rate, being zero in the case of the flat universe.

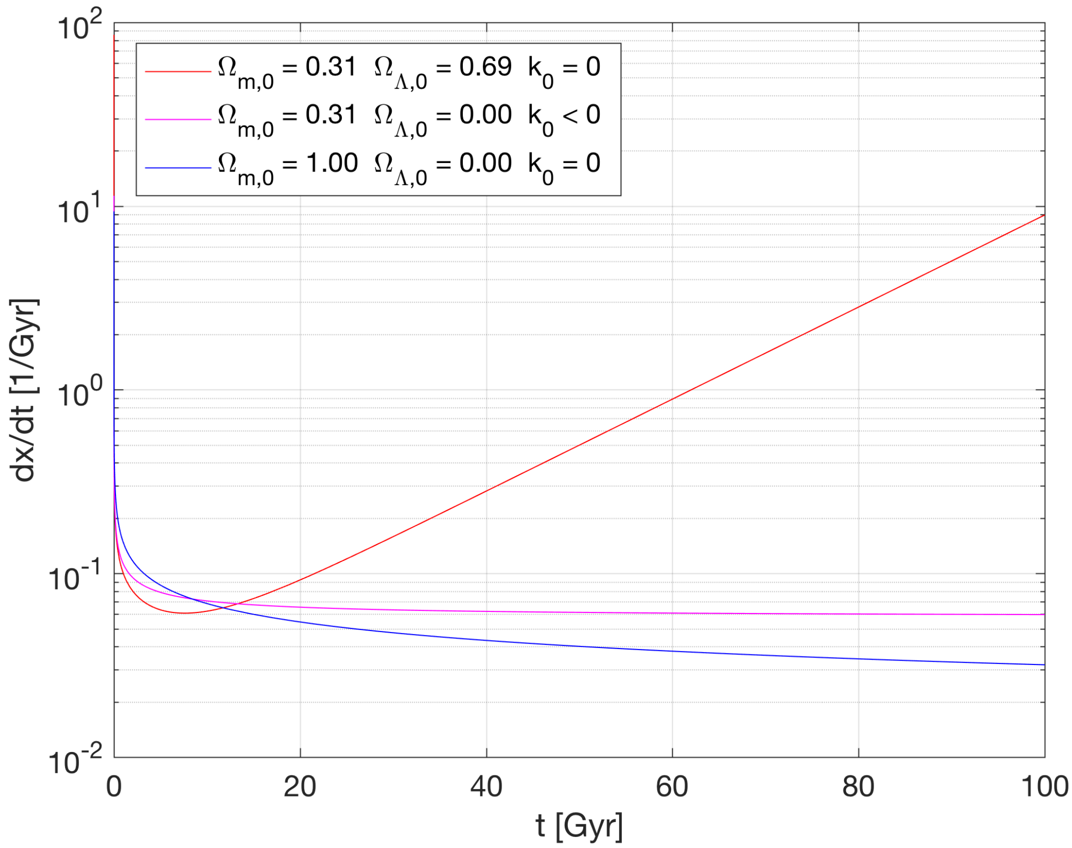

Figure 8: Expansion rate as a function of time for different matter dominated ($\Omega_{r,0} = 0$) models.

9 Conclusions

1. Einstein's general theory of relativity applied to a homogeneous and isotropic universe yields the Friedmann equation, which describes how the scale factor of the universe evolves over time.
2. To solve the Friedmann equation, it needs to be combined with the fluid equation and the equation of state.
3. The Friedmann equation, i.e. the theory of general relativity, requires a cosmological constant to be consistent with cosmological observations, in particular
   - the accelerating expansion of the universe; and
   - the age of the oldest and most distant known galaxy.
4. The exact nature of the cosmological constant is not yet known and it is generally referred to as dark energy. In any case, it counteracts the force of gravity and prevents that the universe collapses back on itself.
5. Beside dark energy, the universe contains a large amount of dark matter of which the exact nature is also not yet known.

10 Further reading

– Andrew Liddle, *An Introduction to Modern Cosmology*, 3rd Edition, John Wiley & Sons Ltd, ISBN 978-1-118-50214-3.

– Barbara Ryden, *Introduction to Cosmology*, 2nd Edition, Cambridge University Press, ISBN 978-1-107-15483-4.

– Alan Guth, *8.286 The Early Universe*, Fall 2013, Massachusetts Institute of Technology: MIT OpenCourseWare (https://ocw.mit.edu)

A MATLAB code

A.1 Age of the universe

A.1.1 Age integral

```
%
%  Code written by Rony Lanssiers in 2017-2018.
%  This work can be shared and adapted as long
%  as appropriate credit is given (CC BY 4.0).
%

function T0 = FCT_Age (H0, Om0, Or0, Ol0)
%
% H0  - Hubble parameter at present time
% Om0 - matter density parameter at present time
% Or0 - radiation density parameter at present time
% Ol0 - lambda density parameter at present time
% T0  - present time, i.e. the age of the universe
%
  Ok0 = 1 - Om0 - Or0 - Ol0;
  T0 = zeros (size (H0));
  for i = 1:numel (H0)
    fct = @(x) x ./ (H0(i) .* sqrt (Or0 + Om0 .* x + Ok0 .* x.^2 + Ol0 .* x.^4));
    T0(i) = integral (fct, 0, 1);
  end
end
```

Listing 1: FCT_Age.m

A.1.2 Age plot

```
%
%  Code written by Rony Lanssiers in 2017-2018.
%  This work can be shared and adapted as long
%  as appropriate credit is given (CC BY 4.0).
%

clear variables;

% 1 Mpc = 3.08567758 * 10^19 km
% 1 Gyr = 3600 * 24 * 365 * 10^9 s
H0Conv = (3600 * 24 * 365 * 10^9) / (3.08567758 * 10^19);

[filename, pathname] = uigetfile ('*.m', 'Select a model definition file.');
run (filename);

H0 = linspace (40, 100, 30);
```

```
for m = 1:numel (model)
  T0 = FCT_Age (H0Conv .* H0, model(m).Om0, model(m).Or0, model(m).Ol0);
  switch sign (-1 * (1 - model(m).Om0 - model(m).Or0 - model(m).Ol0))
    case -1
      kstr = 'k_0 < 0';
    case +1
      kstr = 'k_0 > 0';
    otherwise
      kstr = 'k_0 = 0';
  end
  plotname = sprintf ('\\Omega_{m,0} = %.1f  \\Omega_{\\Lambda,0} = %.1f  %s', model(m).Om0, model(m).Ol0, kstr);
  plot (H0, T0, model(m).line, 'DisplayName', plotname);
  hold on;
end

grid on;
set (gca, 'FontSize', 14);
axis ([40 100 5 25]);
xlabel ('H_0 [km/s/Mpc]');
ylabel ('t_0 [Gyr]');
legend ('show');

plot (67.74, 13.799, 'k*', 'HandleVisibility','off');
set (text (67.74, 13.799 + 0.5, 'Planck'), 'Rotation', 90);

plot (70.0, 13.74, 'k*', 'HandleVisibility','off');
set (text (70.0, 13.74 + 0.5, 'WMAP'), 'Rotation', 90);

print (['age' fbasename], '-dpng', '-r300');
```

Listing 2: PLT_AgeVsHubbleConstant.m

A.1.3 Matter dominated models without cosmological constant

```
%
%  Code written by Rony Lanssiers in 2017-2018.
%  This work can be shared and adapted as long
%  as appropriate credit is given (CC BY 4.0).
%

model(1).Om0 = 0.1;
model(1).Or0 = 0.0;
model(1).Ol0 = 0.0;
model(1).line = 'r';
model(1).color = 'r';

model(2).Om0 = 0.4;
model(2).Or0 = 0.0;
model(2).Ol0 = 0.0;
model(2).line = 'm';
model(2).color = 'm';

model(3).Om0 = 1.0;
model(3).Or0 = 0.0;
model(3).Ol0 = 0.0;
model(3).line = 'b';
model(3).color = 'b';

model(4).Om0 = 1.9;
model(4).Or0 = 0.0;
model(4).Ol0 = 0.0;
model(4).line = 'k';
model(4).color = 'k';

fbasename = 'mattercurvature';
```

Listing 3: DEF_MatterCurvature.m

A.1.4 Matter dominated models with flat geometry

```
%
%  Code written by Rony Lanssiers in 2017-2018.
%  This work can be shared and adapted as long
%  as appropriate credit is given (CC BY 4.0).
%

model(1).Om0 = 0.1;
model(1).Or0 = 0.0;
model(1).Ol0 = 0.9;
model(1).line = 'r';
model(1).color = 'r';

model(2).Om0 = 0.3;
model(2).Or0 = 0.0;
model(2).Ol0 = 0.7;
model(2).line = 'm';
model(2).color = 'm';

model(3).Om0 = 0.5;
model(3).Or0 = 0.0;
model(3).Ol0 = 0.5;
model(3).line = 'b';
model(3).color = 'b';

model(4).Om0 = 1.0;
model(4).Or0 = 0.0;
model(4).Ol0 = 0.0;
model(4).line = 'k';
model(4).color = 'k';

fbasename = 'matterlambda';
```

Listing 4: DEF_MatterLambda.m

A.2 Expansion of the universe

A.2.1 Acceleration equation

```
%
%  Code written by Rony Lanssiers in 2017-2018.
%  This work can be shared and adapted as long
%  as appropriate credit is given (CC BY 4.0).
%

function dXdt = ODE_Acceleration (t, X, H0, Om0, Or0, Ol0)
%
% t       - time
% X(1)    - relative scale factor
% X(2)    - first order time derivative of the relative scale factor
% H0      - Hubble parameter at present time
% Om0     - matter density parameter at present time
% Or0     - radiation density parameter at present time
% Ol0     - lambda density parameter at present time
% dXdt(1) - first order time derivative of the relative scale factor
% dXdt(2) - second order time derivative of the relative scale factor
%
  dXdt = zeros (size (X));
  dXdt(1) = X(2);
  dXdt(2) = - H0^2 * (Or0 / X(1)^3 + 0.5 * Om0 / X(1)^2 - Ol0 * X(1));
end
```

Listing 5: ODE_Acceleration.m

A.2.2 Relative scale factor plot

```
%
%  Code written by Rony Lanssiers in 2017-2018.
%  This work can be shared and adapted as long
%  as appropriate credit is given (CC BY 4.0).
%

clear variables;

H0 = 68; % km/s/Mpc

% 1 Mpc = 3.08567758 * 10^19 km
% 1 Gyr = 3600 * 24 * 365 * 10^9 s
H0Conv = (3600 * 24 * 365 * 10^9) / (3.08567758 * 10^19);

H0InvGyr = H0 * H0Conv; % 1/Gyr

DEF_MatterLambdaCurvature;

for m = 1:numel (model)
  t0 = FCT_Age (H0InvGyr, model(m).Om0, model(m).Or0, model(m).Ol0);

  syseqn = @(t, X) ODE_Acceleration (t, X, H0InvGyr, model(m).Om0, model(m).Or0, model(m).Ol0);

  timespan = [t0 40];
  X0 = [1 H0InvGyr];
  [tfwd, Xfwd] = ode45 (syseqn, timespan, X0);

  timespan = [t0 0];
  X0 = [1 H0InvGyr];
  [tbwd, Xbwd] = ode45 (syseqn, timespan, X0);
  tbwd = flip (tbwd);
  Xbwd = flip (Xbwd);

  switch sign (-1 * (1 - model(m).Om0 - model(m).Or0 - model(m).Ol0))
    case -1
      kstr = 'k_0 < 0';
    case +1
      kstr = 'k_0 > 0';
    otherwise
      kstr = 'k_0 = 0';
  end
  plotname = sprintf ('\\Omega_{m,0} = %.2f  \\Omega_{\\Lambda,0} = %.2f  %s', model(m).Om0, model(m)
    .Ol0, kstr);
  plot ([tbwd; tfwd], [Xbwd(:,1); Xfwd(:,1)], model(m).line, 'DisplayName', plotname);
  hold on;
  plot (t0, 1, [model(m).color 'o'], 'HandleVisibility', 'off');
  set (text (t0, 1.5, sprintf ('%.1f Gyr', t0), 'Color', model(m).color), 'Rotation', 90);
end

grid on;
set (gca, 'FontSize', 14);
xlabel ('t [Gyr]');
ylabel ('x [-]');
legend ('show', 'Location', 'northwest');

print ('scalefactor', '-dpng', '-r300');
```

Listing 6: PLT_ScaleFactorVsTime.m

A.2.3 Matter dominated models

```
%
%  Code written by Rony Lanssiers in 2017-2018.
%  This work can be shared and adapted as long
```

```
%  as appropriate credit is given (CC BY 4.0).
%

model(1).Om0 = 0.31;
model(1).Or0 = 0.0;
model(1).Ol0 = 0.69;
model(1).line = 'r';
model(1).color = 'r';

model(2).Om0 = 0.31;
model(2).Or0 = 0.0;
model(2).Ol0 = 0.0;
model(2).line = 'm';
model(2).color = 'm';

model(3).Om0 = 1.0;
model(3).Or0 = 0.0;
model(3).Ol0 = 0.0;
model(3).line = 'b';
model(3).color = 'b';

model(4).Om0 = 5.0;
model(4).Or0 = 0.0;
model(4).Ol0 = 0.0;
model(4).line = 'k';
model(4).color = 'k';

fbasename = 'matterlambdacurvature';
```

Listing 7: DEF_MatterLambdaCurvature.m