Cosmological Distances

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Abstract

Distances in the universe are not as easy to define as the distances in Euclidean space people are familiar with. It is shown how the geometry and expansion of the universe affects distances measures, leading to rather peculiar effects.

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1 Angular-diameter distance

1.1 Definition

One of the forms of the Friedmann-Lemaître-Robertson-Walker metric for a homogeneous and isotropic universe is given by:

$$ds^2 = c^2\,dt^2 - a^2(t)\left[dr^2 + S_\kappa^2(r)\,d\Omega^2\right] \tag{1}$$

$$\text{with}\quad d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2$$

$$\text{and}\quad S_\kappa(r) = \begin{cases} R_0\,\sin\left(\frac{r}{R_0}\right) & \text{when} \quad \kappa = +1 \\ r & \text{when} \quad \kappa = 0 \\ R_0\,\sinh\left(\frac{r}{R_0}\right) & \text{when} \quad \kappa = -1 \end{cases}$$

Consider an object with length l along a circle with radius r, centred around the origin of a coordinate system. In that case, the comoving radial coordinate r does not change over the length of interest and hence $dr^2 = 0$. If additionally, the coordinate system's orientation is chosen such that the circle lies in the plane for which $\phi = 0$, then the spatial part of the metric simplifies to:

$$dl^2 = a^2(t)\, S_\kappa^2(r)\, d\theta^2$$

Furthermore:

$$dl = a(t)\, S_\kappa(r)\, d\theta$$

$$l = a(t)\, S_\kappa(r)\, \theta$$

$$\theta = \frac{l}{\underbrace{a(t)\, S_\kappa(r)}_{=d_A}}$$

The quantity d_A is called the angular-diameter distance. It represents the distance at which an object with length l needs to be in a Euclidean non-expanding space to observe it with an apparent angular size θ.

Indeed, consider an object with length l, located at a distance D from an observer in a Euclidean non-expanding space. The relationship between length, distance and apparent angular size θ as seen by the observer, results from simple trigonometry:

$$l = 2\, D\, \tan\left(\frac{\theta}{2}\right)$$

If the object's length is much smaller than its distance, i.e. $l << D$, a good approximation is:

$$l \approx D\, \theta$$

Consequently, the object's apparent angular size is approximately:

$$\theta \approx \frac{l}{D}$$

A drawback for computing the angular-diameter distance is that $a(t)\, S_\kappa(r)$ is an unobservable quantity. What is needed is an expression based on something that is measurable such as the cosmological redshift z. Given the definition of $S_\kappa(r)$, this requires a closer look at the radial distance r and the present radius of curvature R_0.

1.2 Radial distance

Remember[1] the expansion-redshift law which relates the scale factor a to the cosmological redshift z:

$$z + 1 = \frac{a_0}{a} \tag{2}$$

[1] Rony Lanssiers, *Foundations of Modern Cosmology*

Differentiating it yields:

$$-\frac{a_0\,da}{a^2} = dz \tag{3}$$

Light waves always follow a null geodesic which means for FLRW metric (1):

$$ds^2 = 0 = c^2\,dt^2 - a^2(t)\left[dr^2 + S_\kappa^2(r)\,d\Omega^2\right] \tag{4}$$

Light waves travel along a geodesic with constant θ and ϕ to an observer located at the center of the coordinate system. Consequently:

$$S_\kappa^2(r)\,d\Omega^2 = 0 \tag{5}$$

Combining equations (4) and (5) yields:

$$c^2 dt^2 = a^2(t)\,dr^2$$

$$c\,dt = -a(t)\,dr$$

$$\frac{c\,dt}{a(t)} = -dr \tag{6}$$

Recall[2] that in terms of the present density parameters $\Omega_{i,0}$ and the Hubble constant H_0, the Friedmann equation takes the form:

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\left(\frac{a_0}{a}\right)^4\Omega_{r,0} + \left(\frac{a_0}{a}\right)^3\Omega_{m,0} + \left(\frac{a_0}{a}\right)^2\Omega_{k,0} + \Omega_{\Lambda,0}\right] \tag{7}$$

Using expansion-redshift law (2), equation (7) becomes:

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\underbrace{\left[(z+1)^4\,\Omega_{r,0} + (z+1)^3\,\Omega_{m,0} + (z+1)^2\,\Omega_{k,0} + \Omega_{\Lambda,0}\right]}_{\equiv\Omega(z)}$$

$$\dot{a} = \frac{da}{dt} = a\,H_0\,\sqrt{\Omega(z)}$$

$$dt = \frac{da}{a\,H_0\,\sqrt{\Omega(z)}} \tag{8}$$

Substituting equation (3) in equation (8) results in:

[2] Rony Lanssiers, *Numerical Models for the Expanding Universe*

$$dt = -\frac{a\, dz}{a_0\, H_0\, \sqrt{\Omega(z)}} \tag{9}$$

Substituting equation (6) in equation (9) results in:

$$dr = \frac{c\, dz}{a_0\, H_0\, \sqrt{\Omega(z)}} \tag{10}$$

Integrating equation (10) over the range 0 to z gives an expression for the radial distance r based on the cosmological redshift z:

$$r = \int_0^z \frac{c\, dz}{a_0\, H_0\, \sqrt{\Omega(z)}}$$

$$r = \frac{c}{a_0\, H_0} \int_0^z \frac{dz}{\sqrt{\Omega(z)}} \tag{11}$$

1.3 Radius of curvature

Remember[3] that the curvature k based on the present curvature density parameter $\Omega_{k,0}$ is given by:

$$k = -\left(\frac{a_0\, H_0}{c}\right)^2 \Omega_{k,0}$$

Recall[4] that the relationship between curvatures k and κ involves R_0:

$$k = \frac{\kappa}{R_0^2}$$

The present radius of curvature R_0 is then:

$$R_0 = \frac{1}{\sqrt{|k|}}$$

$$R_0 = \frac{c}{a_0\, H_0 \sqrt{|\Omega_{k,0}|}} \tag{12}$$

1.4 Large faint reddish objects

The values of r and R_0 obtained via respectively equations (11) and (12) allow computing $S_\kappa(r)$ which in turn is needed to obtain d_A. Expansion-redshift law (2) helps to eliminate the scale factor from the final expression. For example, for a positive curvature $(\kappa = +1)$:

[3]Rony Lanssiers, *Numerical Models for the Expanding Universe*
[4]Rony Lanssiers, *The Friedmann-Lemaître-Robertson-Walker Metric*

$$d_A = a\, S_\kappa(r)$$

$$d_A = a\, R_0\, \sin\left(\frac{r}{R_0}\right)$$

$$d_A = a\, \frac{c}{a_0\, H_0 \sqrt{|\Omega_{k,0}|}}\, \sin\left(\frac{\frac{c}{a_0\, H_0} \int_0^z \frac{dz}{\sqrt{\Omega(z)}}}{\frac{c}{a_0\, H_0 \sqrt{|\Omega_{k,0}|}}}\right)$$

$$d_A = \frac{a}{a_0} \frac{c}{H_0 \sqrt{|\Omega_{k,0}|}}\, \sin\left(\sqrt{|\Omega_{k,0}|} \int_0^z \frac{dz}{\sqrt{\Omega(z)}}\right)$$

$$d_A = \frac{c}{(z+1)\ H_0 \sqrt{|\Omega_{k,0}|}}\, \sin\left(\sqrt{|\Omega_{k,0}|} \int_0^z \frac{dz}{\sqrt{\Omega(z)}}\right)$$

For a negative curvature ($\kappa = -1$), the result is similar with the only difference that the sine becomes a hyperbolic sine:

$$d_A = \frac{c}{(z+1)\ H_0 \sqrt{|\Omega_{k,0}|}}\, \sinh\left(\sqrt{|\Omega_{k,0}|} \int_0^z \frac{dz}{\sqrt{\Omega(z)}}\right)$$

In the absence of curvature ($\kappa = 0$):

$$d_A = a\, S_\kappa(r)$$

$$d_A = a\, r$$

$$d_A = \frac{a}{a_0} \frac{c}{H_0} \int_0^z \frac{dz}{\sqrt{\Omega(z)}}$$

$$d_A = \frac{c}{(z+1)\ H_0} \int_0^z \frac{dz}{\sqrt{\Omega(z)}}$$

Figure 1 shows the angular-diameter distance as a function of the cosmological redshift for different geometrically flat models with no radiation contribution.

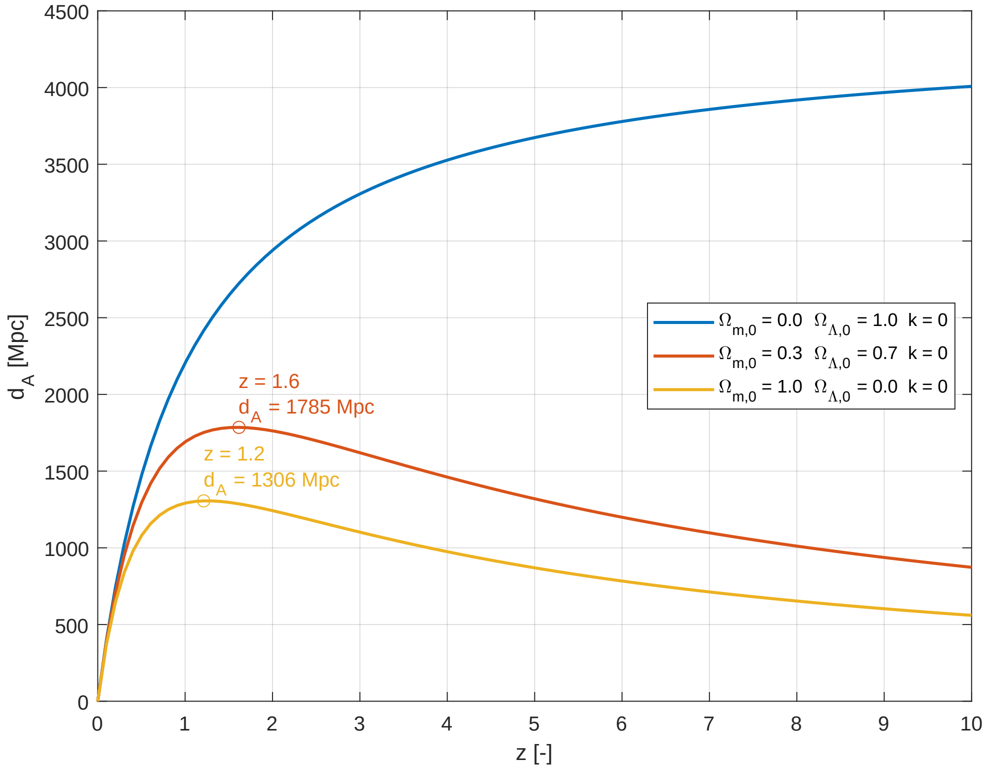

Figure 1: Angular-diameter distance d_A as a function of the cosmological redshift z for different geometrically flat ($k = 0$) models with no radiation contribution ($\Omega_{r,0} = 0$).

Except for the empty universe driven by only a cosmological constant, d_A first increases with increasing z up to a certain maximum, after which a continued decrease sets in. For the standard model, the maximum lies at a cosmological redshift of 1.6 , corresponding with an angular-diameter distance of 1785 Mpc. Simply put:

> If the universe would only contain objects of the same size,
> the night sky would be filled with large faint reddish objects.