Numerical Models for the Expanding Universe

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Abstract

Following an overview of the equations governing the evolution of a homogeneous and isotropic universe, it is shown how to solve the equations both analytically for simplified models and numerically in the most general case. The results are used to illustrate how the evolution and fate of the universe depends on its composition. The age of the universe is calculated as a function of composition and value of the Hubble constant and the model which best matches recent cosmological observations is identified.

The model source code written in the m-language, suitable for both MATLAB[1] and GNU Octave[2], is available via the GitHub repository https://www.github.com/rlanssiers/expandinguniverse.

Contents

[1] https://www.mathworks.com
[2] https://www.gnu.org/software/octave

1 Equations

1.1 The Friedmann equation

Based on Einstein's theory of general relativity, the Russian cosmologist Alexander Friedmann[3] derived the equation which describes how the scale factor[4] $a(t)$ evolves over time for a homogeneous and isotropic universe.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\,\pi\,G}{3\,c^2}\,\epsilon - \frac{k\,c^2}{a^2} + \frac{\Lambda}{3} \tag{1}$$

The Friedmann equation is a first order differential equation in which the scale factor $a(t)$ and the energy density $\epsilon(t)$ are time dependent variables. All the other parts are constants with G being the gravitational constant and c the speed of light. The factor k is a dimensionless number and describes the curvature of space. Depending on whether it is negative, zero or positive, space has a hyperbolic, flat or spherical geometry.

The term Λ in Friedmann's equation is the cosmological constant which Einstein added to the field equations of general relativity after he had realized that otherwise, all solutions were describing a universe which is expanding or contracting at any given moment in time. Such a dynamic universe could not be reconciled with widespread belief at that time that the universe was static. The introduction of a cosmological constant turned out to be necessary to achieve a static solution.

When Vesto Slipher[5] and Edwin Hubble[6] discovered that the universe is not static but expanding, Einstein dropped the cosmological constant again and reverted to his original field equations. This remained unchanged until 1998 when new observational evidence showed that the expansion of the universe is accelerating, something which again required a cosmological constant to be explainable. Although its exact nature is still unknown, whatever the cosmological constant represents is now widely accepted as an important component of our universe.

It is convenient to define a constant energy density ϵ_Λ related to the cosmological constant as:

$$\epsilon_\Lambda \equiv \frac{c^2}{8\,\pi\,G}\,\Lambda \tag{2}$$

Using that constant energy density, the cosmological constant term of the Friedmann equation becomes:

$$\frac{\Lambda}{3} = \frac{8\,\pi\,G}{3\,c^2}\,\epsilon_\Lambda$$

[3] Alexander Friedmann was a Russian physicist and mathematician who lived from 1888 till 1925.
[4] The scale factor is a measure of how the unit vectors of a co-moving coordinate system scale with time.
[5] Vesto Slipher was an American astronomer who lived from 1875 till 1969.
[6] Edwin Hubble was an American astronomer who lived from 1889 till 1953.

If further distinction is made between the energy density contributions ϵ_r from radiation and ϵ_m from matter, the Friedmann equation takes the form:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\,\pi\,G}{3\,c^2}\,(\epsilon_r + \epsilon_m + \epsilon_\Lambda) - \frac{k\,c^2}{a^2}$$

Note that the left hand side ratio in the Friedmann equation actually equals the square of the Hubble parameter[7] H, given that:

$$\boxed{H = \frac{\dot{a}}{a}} \tag{3}$$

1.2 The fluid equation

The volume $V(t)$ of a sphere with radius $r(t) = a(t)\,r_0$ is:

$$V(t) = \frac{4}{3}\,\pi\,a(t)^3\,r_0^3$$

Its time derivative is consequently:

$$\frac{dV}{dt} = \frac{4}{3}\,\pi\,r_0^3\,3\,a^2\,\frac{da}{dt}$$

$$\frac{dV}{dt} = 3\,V\,\frac{\dot{a}}{a} \tag{4}$$

The internal energy $E(t)$ of a sphere with volume $V(t)$ and energy density $\epsilon(t)$ is:

$$E(t) = V(t)\,\epsilon(t)$$

Its time derivative is consequently:

$$\frac{dE}{dt} = V\,\frac{d\epsilon}{dt} + \epsilon\,\frac{dV}{dt}$$

$$\frac{dE}{dt} = V\,\dot{\epsilon} + \epsilon\,3\,V\,\frac{\dot{a}}{a}$$

$$\frac{dE}{dt} = V\left(\dot{\epsilon} + 3\,\frac{\dot{a}}{a}\,\epsilon\right) \tag{5}$$

The first law of thermodynamics states that the change in internal energy dE of a system expanding in

[7]The Hubble parameter is the proportionality factor in Hubble's law which relates the recession velocity of distant objects to their distance.

a quasi-static process is the sum of the amount of heat dQ supplied to the system and the work done on the system $-P\,dV$:

$$dE = dQ - P\,dV$$

$$dQ = dE + P\,dV \tag{6}$$

In a homogeneous universe there is no heat transfer in or out of a co-moving volume (adiabatic expansion). Combining equations (4), (5) and (6) then yields:

$$dQ = 0$$

$$dE + P\,dV = 0$$

$$\frac{dE}{dt} + P\,\frac{dV}{dt} = 0$$

$$V\left(\dot{\epsilon} + 3\,\frac{\dot{a}}{a}\,\epsilon\right) + P\,3\,V\,\frac{\dot{a}}{a} = 0$$

$$\boxed{\dot{\epsilon} + 3\,\frac{\dot{a}}{a}\,(\epsilon + P) = 0} \tag{7}$$

A slightly different way to write equation (7) is given by:

$$\dot{\epsilon}\,\frac{a}{\dot{a}} = -3\,(\epsilon + P)$$

1.3 The acceleration equation

The acceleration equation is not a new equation but rather a combination of the Friedmann equation and the fluid equation. Taking the time derivative of Friedmann equation (1), dividing by $2a\dot{a}$ and substituting the fluid equation gives:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\,\pi\,G}{3\,c^2}\,\epsilon - \frac{k\,c^2}{a^2} + \frac{\Lambda}{3}$$

$$\dot{a}^2 = \frac{8\,\pi\,G}{3\,c^2}\,\epsilon\,a^2 - k\,c^2 + \frac{\Lambda\,a^2}{3}$$

$$2\,\dot{a}\,\ddot{a} = \frac{8\,\pi\,G}{3\,c^2}\,(\dot{\epsilon}\,a^2 + 2\,\epsilon\,a\,\dot{a}) + \frac{2\,a\,\dot{a}\,\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = \frac{4\,\pi\,G}{3\,c^2}\left(\dot{\epsilon}\,\frac{a}{\dot{a}} + 2\,\epsilon\right) + \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = \frac{4\,\pi\,G}{3\,c^2}\,(-3\,\epsilon - 3\,P + 2\,\epsilon) + \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = \frac{-4\,\pi\,G}{3\,c^2}\,(\epsilon + 3\,P) + \frac{\Lambda}{3}$$

The acceleration equation tells something about how rapidly a changes over time. If there would be no cosmological constant Λ in the acceleration equation, its outcome would never be positive, taking into account that the energy density ϵ and pressure P of matter and radiation are non-negative quantities. This would imply that the universe expands at a constant or decreasing rate, but both the Supernova Cosmology Project [8] and the High-Z Supernova Search Team [9] independently discovered in 1998 that the expansion of the universe is actually accelerating. The cosmological constant is the only possibility to achieve a positive value as outcome of the acceleration equation and explain the observational evidence for an accelerating expansion.

1.4 The equation of state

1.4.1 General form

For the kind of substances which are playing a role in cosmological calculations, the equation of state is a linear relationship between pressure $P(t)$ and energy density $\epsilon(t)$:

$$P = w\,\epsilon \tag{8}$$

It is shown in the appendix that the equation of state for a particle gas satisfies this general form, with a proportionality factor w determined by the average speed of the particles:

$$w = \frac{1}{3}\,\frac{\bar{v}^2}{c^2} \tag{9}$$

Based on equation (8) and the calculation rules for derivatives, equation (7) transforms into:

$$\dot{\epsilon} + 3\,\frac{\dot{a}}{a}\,(\epsilon + w\,\epsilon) = 0$$

$$\dot{\epsilon} + 3\,(1+w)\,\frac{\dot{a}}{a}\,\epsilon = 0$$

$$\frac{1}{a^{3\,(1+w)}}\,\frac{d}{dt}\left(\epsilon\,a^{3\,(1+w)}\right) = 0$$

$$\frac{d}{dt}\left(\epsilon\,a^{3\,(1+w)}\right) = 0$$

$$\epsilon\,a^{3\,(1+w)} = cst$$

[8] http://supernova.lbl.gov

[9] https://www.cfa.harvard.edu/supernova/home.html

$$\epsilon \propto \frac{1}{a^{3\,(1+w)}} \tag{10}$$

1.4.2 Radiation

In cosmology, radiation is the term used for massless particles moving at speeds close to the speed of light. Photons are the obvious example of such particles but in a cosmological context, neutrinos are also counted as a form of radiation. Recent research has shown that neutrinos might have a rest mass but still sufficiently small to consider them as massless particles without invalidating the theory.

For radiation, $\bar{v} = c$ and consequently, based on equations (8) and (9):

$$\boxed{\begin{aligned} w_r &= \frac{1}{3} \\ P_r &= \frac{1}{3}\,\epsilon_r \end{aligned}}$$

The proportionality of expression (10) then becomes:

$$\epsilon_r \propto \frac{1}{a^4}$$

Denoting the present value of the scale factor as a_0 and the present value of the radiation energy density as $\epsilon_{r,0}$ allows writing the proportionality as an equality:

$$\epsilon_r = \left(\frac{a_0}{a}\right)^4 \epsilon_{r,0} \tag{11}$$

1.4.3 Non-relativistic matter

In cosmology, non-relativistic matter is the term used for massive particles moving at speeds considerably less than the speed of light. On Earth, all matter is made of protons and neutrons and called baryonic [10] matter. Several research groups have shown that the estimated amount of baryonic matter in the universe is not always in agreement with the expected amount of matter based on particular observations. For example, the constant rotational velocity of galaxies at distances far away from their centre is unexplainable by their baryonic content alone. Gravitational lensing is another phenomenon which indicates there is vastly more matter in the universe than we are able to see. This unknown non-baryonic matter is commonly called dark matter, given that it is hidden from direct observation with today's means. Recent estimates yield a baryonic matter contribution to $\Omega_{m,0}$ of 4.8% and a dark matter contribution of 26.2%.

For non-relativistic matter, $\bar{v}^2 << c^2$ and consequently, based on equations (8) and (9):

[10] Formally, baryons are particles formed by 3 quarks unlike mesons which consist of 2 quarks.

$$w_m \approx 0$$

$$P_m \approx 0$$

The proportionality of expression (10) then becomes:

$$\epsilon_m \propto \frac{1}{a^3}$$

Denoting the present value of the scale factor as a_0 and the present value of the matter energy density as $\epsilon_{m,0}$ allows writing the proportionality as an equality:

$$\epsilon_m = \left(\frac{a_0}{a}\right)^3 \epsilon_{m,0} \tag{12}$$

1.4.4 Dark energy

Applying fluid equation (7) to the cosmological constant gives:

$$\dot{\epsilon_\Lambda} + 3\,\frac{\dot{a}}{a}\,(\epsilon_\Lambda + P_\Lambda) = 0$$

As ϵ_Λ is a constant by definition, its time derivative $\dot{\epsilon_\Lambda}$ is zero and consequently:

$$3\,\frac{\dot{a}}{a}\,(\epsilon_\Lambda + P_\Lambda) = 0$$

$$\epsilon_\Lambda + P_\Lambda = 0$$

This shows that the cosmological constant represents something which creates a negative pressure:

$$w_\Lambda = -1$$

$$P_\Lambda = -\epsilon_\Lambda$$

Whatever it may be, that something is generally referred to as dark energy and its true nature is one of the biggest mysteries of modern physics. Not only does it create a negative pressure, it appears to violate the general law of energy conservation, given that its energy density ϵ_Λ as defined in equation (2) remains constant as the universe expands. Its value at the present time equals its value at any other time:

$$\epsilon_\Lambda = \epsilon_{\Lambda,0} \tag{13}$$

2 Solutions

2.1 Analytical solutions

Analytical solutions for the Friedmann equation only exist for a number of simplified cases. In its most general form, solutions are obtained by making use of numerical methods. Even though they do not describe the full picture, examining some of the simplified analytical solution helps to understand the different era the universe went through during its evolution.

The Friedmann equation written with distinct energy density contributions is given by:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\,\pi\,G}{3\,c^2}\,(\epsilon_r + \epsilon_m + \epsilon_\Lambda) - \frac{k\,c^2}{a^2}$$

Observational evidence indicates that the universe has a flat geometry or at least one which is very close to flat. The first simplification therefore consists of taking $k = 0$.

Next, consider a universe which only contains radiation, i.e. $\epsilon_m = \epsilon_\Lambda = 0$. With ϵ_r given by equation (11), the Friedman equation then simplifies to:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\,\pi\,G}{3\,c^2}\left(\frac{a_0}{a}\right)^4 \epsilon_{r,0}$$

$$a\,\dot{a} = constant$$

$$\frac{d}{dt}\left(a^2\right) = constant$$

$$a^2 \propto t$$

$$a \propto t^{\frac{1}{2}}$$

$$a = a_0\left(\frac{t}{t_0}\right)^{\frac{1}{2}}$$

In a similar fashion, the case of a universe which only contains matter, i.e. $\epsilon_r = \epsilon_\Lambda = 0$, and with ϵ_m given by equation (12) leads to:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\,\pi\,G}{3\,c^2}\left(\frac{a_0}{a}\right)^3 \epsilon_{m,0}$$

$$a^{\frac{1}{2}}\,\dot{a} = constant$$

$$\frac{d}{dt}\left(a^{\frac{3}{2}}\right) = constant$$

$$a^{\frac{3}{2}} \propto t$$

$$a \propto t^{\frac{2}{3}}$$

$$a = a_0 \left(\frac{t}{t_0} \right)^{\frac{2}{3}}$$

Finally, the case of an empty universe which is only driven by a constant dark energy density as given in equation (13) results in:

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\,\pi\,G}{3\,c^2}\,\epsilon_{\Lambda,0}$$

$$\frac{\dot{a}}{a} = constant$$

Remembering equation (3), the above conclusion that the ratio $\dot{a}/a$ is constant allows to write:

$$\frac{\dot{a}}{a} = H_0$$

$$\frac{d}{dt}(\ln a) = H_0$$

$$\ln a = H_0\,t + constant$$

$$a = e^{H_0\,t + constant} \tag{14}$$

This is obviously also true at the present time t_0:

$$a_0 = e^{H_0\,t_0 + constant} \tag{15}$$

Dividing equation (14) by equation (15) eliminates the unknown constant and results in:

$$a = a_0\,e^{H_0\,(t - t_0)}$$

The solutions for the Friedmann equation for universes with respectively only radiation, matter or dark energy are shown in figure 1.

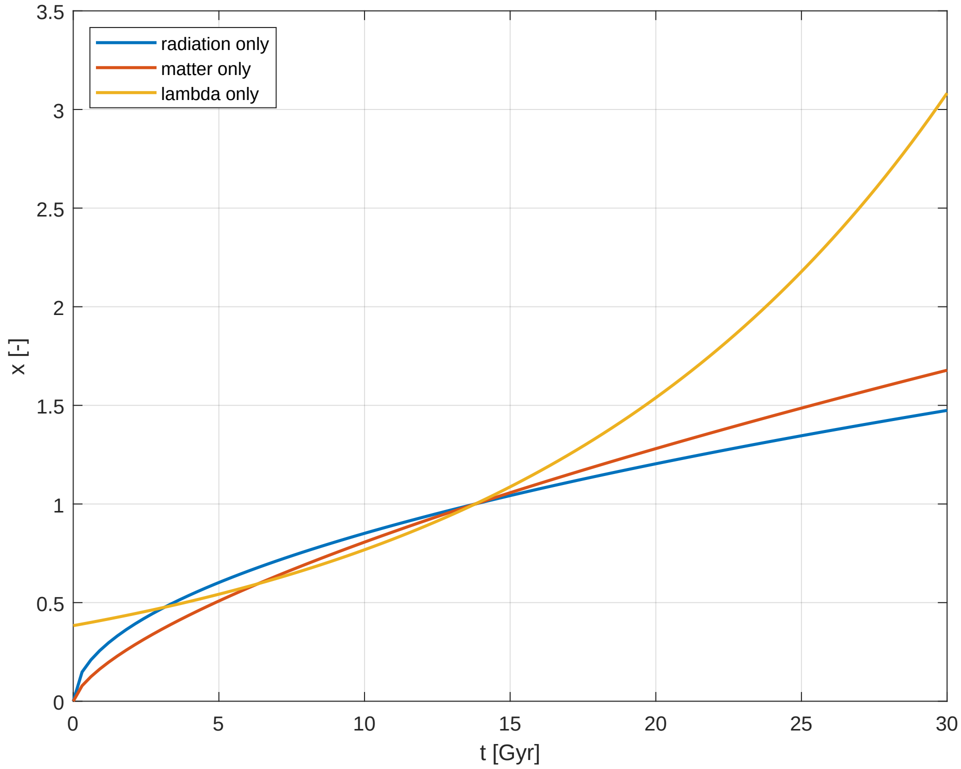

Figure 1: Relative scale factor $x = a/a_0$ as a function of time for universes with a present age of $13.8\ \mathrm{Gyr}$ containing only radiation, matter or dark energy.

The universe has evolved from an era during which radiation was dominant, over a matter dominated period, to the present dark energy phase. The curves in figure 1 illustrate why the long term expansion of the universe was first driven by radiation, then by matter and finally by dark energy and why the expansion is observed to be accelerating.

2.2 Numerical solutions

The energy density that the universe without cosmological constant needs to have to be flat is called the critical energy density $\epsilon_c(t)$ and is defined as:

$$\boxed{\epsilon_c \equiv \frac{3\,c^2}{8\,\pi\,G} H^2} \tag{16}$$

Substituting the critical energy density (16) in Friedmann equation (1) shows that the curvature k must indeed be zero at that density (if $\Lambda = 0$).

Converted to units of mass, it turns out that the equivalent[11] mass density $\rho_c(t)$ is very small and about

[11] Remember from Einstein's theory of special relativity that $E = m\,c^2$.

the same as the mass of 6 protons per cubic meter:

$$\rho_c \approx 10^{-26}\ \mathrm{kg\ m^{-3}}$$

$$\rho_c \approx 6\ \mathrm{m_p\ m^{-3}}$$

The critical energy density $\epsilon_c(t)$ is useful to remove absolute energy densities from the equations and to replace them with dimensionless density parameters $\Omega(t)$ defined as:

$$\boxed{\Omega \equiv \frac{\epsilon}{\epsilon_c}} \tag{17}$$

Combining equations (16) and (17) yields an expression for the energy density as a function of the density parameter (and the Hubble parameter):

$$\boxed{\epsilon = \frac{3\, c^2\, H^2}{8\, \pi\, G}\, \Omega} \tag{18}$$

Applying equation (18) at the present time for respectively radiation, non-relativistic matter and the cosmological constant gives:

$$\begin{cases} \epsilon_{r,0} = \dfrac{3\, c^2\, H_0^2}{8\, \pi\, G}\, \Omega_{r,0} & \text{(19a)} \\[2ex] \epsilon_{m,0} = \dfrac{3\, c^2\, H_0^2}{8\, \pi\, G}\, \Omega_{m,0} & \text{(19b)} \\[2ex] \epsilon_{\Lambda,0} = \dfrac{3\, c^2\, H_0^2}{8\, \pi\, G}\, \Omega_{\Lambda,0} & \text{(19c)} \end{cases}$$

Substituting equation (19a) in equation (11), the radiation energy density $\epsilon_r(t)$ becomes:

$$\epsilon_r = \left(\frac{a_0}{a}\right)^4 \frac{3\, c^2\, H_0^2}{8\, \pi\, G}\, \Omega_{r,0} \tag{20}$$

Substituting equation (19b) in equation (12), the non-relativistic matter energy density $\epsilon_m(t)$ becomes:

$$\epsilon_m = \left(\frac{a_0}{a}\right)^3 \frac{3\, c^2\, H_0^2}{8\, \pi\, G}\, \Omega_{m,0} \tag{21}$$

Substituting equation (19c) in equation (13), the cosmological constant energy density ϵ_Λ becomes:

$$\epsilon_\Lambda = \frac{3\,c^2\,H_0^2}{8\,\pi\,G}\,\Omega_{\Lambda,0} \tag{22}$$

It is convenient to define the present curvature density parameter $\Omega_{k,0}$ as:

$$\boxed{\Omega_{k,0} \equiv -\frac{k\,c^2}{a_0^2\,H_0^2}}$$

The curvature term of Friedmann equation (1) then becomes:

$$-\frac{k\,c^2}{a^2} = H_0^2\left(\frac{a_0}{a}\right)^2\Omega_{k,0} \tag{23}$$

Substituting equations (20), (21), (22) and (23) in Friedmann equation (1) yields:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\,\pi\,G}{3\,c^2}\,(\epsilon_r + \epsilon_m + \epsilon_\Lambda) - \frac{k\,c^2}{a^2}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\,\pi\,G}{3\,c^2}\left[\frac{3\,c^2\,H_0^2}{8\,\pi\,G}\left(\frac{a_0}{a}\right)^4\Omega_{r,0} + \frac{3\,c^2\,H_0^2}{8\,\pi\,G}\left(\frac{a_0}{a}\right)^3\Omega_{m,0} + \frac{3\,c^2\,H_0^2}{8\,\pi\,G}\,\Omega_{\Lambda,0}\right] + H_0^2\left(\frac{a_0}{a}\right)^2\Omega_{k,0}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\left(\frac{a_0}{a}\right)^4\Omega_{r,0} + \left(\frac{a_0}{a}\right)^3\Omega_{m,0} + \left(\frac{a_0}{a}\right)^2\Omega_{k,0} + \Omega_{\Lambda,0}\right] \tag{24}$$

The relative scale factor $x(t)$ is defined as the ratio between the scale factor $a(t)$ and the scale factor at the present time $a(t_0)$:

$$x \equiv \frac{a}{a_0} \tag{25}$$

The first order time derivative of the relative scale factor is then:

$$\dot{x} = \frac{\dot{a}}{a_0}$$

Note that equation (3) results in a similar equation in terms of the relative scale factor:

$$H = \frac{\dot{a}}{a} = \frac{\dot{x}\,a_0}{x\,a_0} = \frac{\dot{x}}{x}$$

Changing over to the relative scale factor $x(t)$ in equation (24) yields:

$$\left(\frac{\dot{x}}{x}\right)^2 = H_0^2 \left(\frac{\Omega_{r,0}}{x^4} + \frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{k,0}}{x^2} + \Omega_{\Lambda,0}\right) \tag{26}$$

Taking into account that the relative scale factor at the present time $x(t_0)$ equals 1 and that $\dot{x}(t_0)/x(t_0)$ equals the Hubble constant H_0, equation (26) applied to the present time t_0 results in an important constraint on the sum of all density parameters:

$$H_0^2 = H_0^2 \left(\Omega_{r,0} + \Omega_{m,0} + \Omega_{k,0} + \Omega_{\Lambda,0}\right)$$

$$\boxed{1 = \Omega_{r,0} + \Omega_{m,0} + \Omega_{k,0} + \Omega_{\Lambda,0}} \tag{27}$$

Eliminating x^2 from the denominators of equation (26) gives:

$$\frac{\dot{x}^2}{x^2} = \frac{H_0^2}{x^2} \left(\frac{\Omega_{r,0}}{x^2} + \frac{\Omega_{m,0}}{x} + \Omega_{k,0} + \Omega_{\Lambda,0}\, x^2\right)$$

$$\dot{x} = H_0 \sqrt{\frac{\Omega_{r,0}}{x^2} + \frac{\Omega_{m,0}}{x} + \Omega_{k,0} + \Omega_{\Lambda,0}\, x^2}$$

Incorporating the constraint from equation (27) for the sum of the density parameters into this expression leads to:

$$\boxed{\dot{x} = H_0 \sqrt{\frac{\Omega_{r,0}}{x^2} + \frac{\Omega_{m,0}}{x} + \Omega_{\Lambda,0}\, x^2 + 1 - \Omega_{r,0} - \Omega_{m,0} - \Omega_{\Lambda,0}}} \tag{28}$$

This is a first order differential equation which has no analytical solution it its most general form given above. The square root makes it difficult to solve the equation numerically as the square root yields a complex number when its argument is negative.

One way to eliminate the squares is by differentiating the left and right hand side of the expression for $\dot{x}^2$ and then dividing both sides by $2\,\dot{x}$:

$$\dot{x}^2 = H_0^2 \left(\frac{\Omega_{r,0}}{x^2} + \frac{\Omega_{m,0}}{x} + \Omega_{k,0} + \Omega_{\Lambda,0}\, x^2\right)$$

$$2\,\dot{x}\,\ddot{x} = H_0^2 \left(-2\,\dot{x}\,\frac{\Omega_{r,0}}{x^3} - \dot{x}\,\frac{\Omega_{m,0}}{x^2} + 2\,\dot{x}\,\Omega_{\Lambda,0}\, x\right)$$

$$\ddot{x} = -H_0^2 \left(\frac{\Omega_{r,0}}{x^3} + \frac{\Omega_{m,0}}{2\,x^2} - \Omega_{\Lambda,0}\, x\right)$$

Next, this second order differential equation is converted into a system of 2 first order differential equations:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -H_0^2 \left(\dfrac{\Omega_{r,0}}{x^3} + \dfrac{\Omega_{m,0}}{2\,x^2} - \Omega_{\Lambda,0}\,x \right) \end{cases}$$

Taking $H = \dot{x}/x$ and the definition of the relative scale factor (25) into account, following initial conditions apply:

$$\begin{cases} x(t_0) = 1 \\ \dot{x}(t_0) = H_0 \end{cases}$$

The result of solving the system of differential equations numerically with the aid of modern computer software is shown in figure 2.

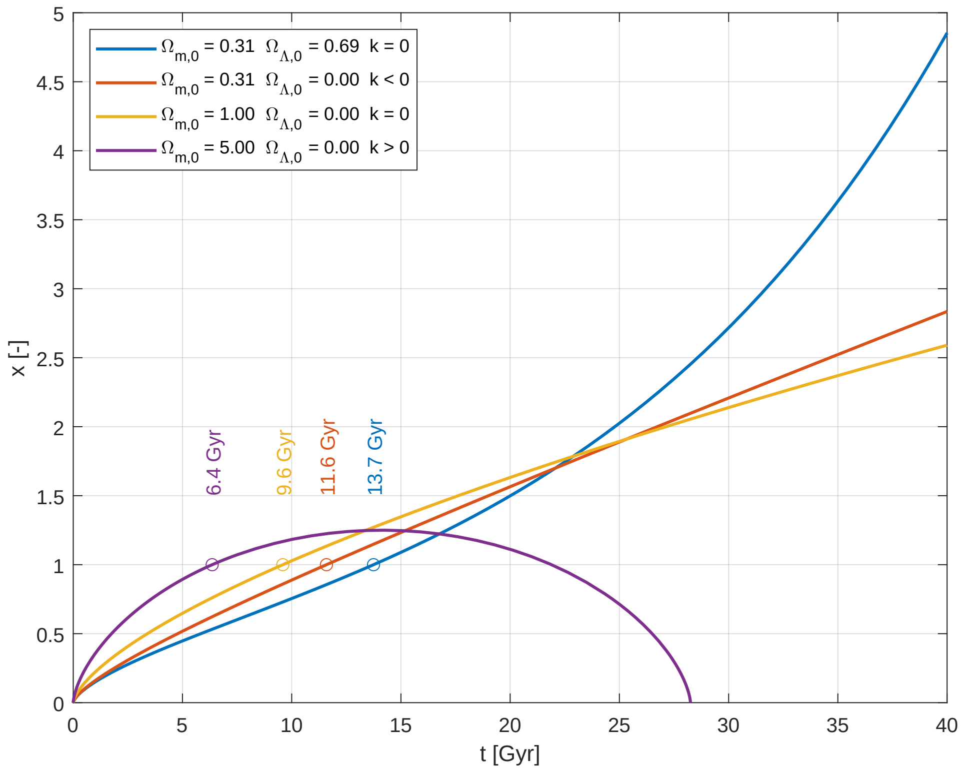

Figure 2: Relative scale factor as a function of time for different matter dominated ($\Omega_{r,0} = 0$) models.

The blue curve combines (cold dark) matter and a cosmological constant with a flat or Euclidean geometry. It is commonly denoted as the ΛCDM model and best fits the observations. According to this benchmark model, the present universe is 13.7 Gyr old. Its expansion accelerates indefinitely and will eventually affect space at small scales and even rip atoms apart. Whether the universe will really end in such a Big Rip remains an open question as nobody knows how the cosmological constant or whatever it represents behaves on the long run.

The other models shown in figure 2 are all matter dominated models without cosmological constant, but with different geometries. The orange curve represents a negatively curved universe, i.e. one with hyperbolic geometry. This open universe expands eternally at a rate which reaches a constant value at late times. Its ultimate fate is a big cold void, sometimes called the Big Chill, with all celestial objects in an end-of-life state and separated so far apart that remnant light is unable to bridge the gap between them.

The yellow curve represents a universe with a density which exactly matches the critical density. In absence of a cosmological constant, it consequently has a flat or Euclidean geometry. The expansion of this flat universe continues eternally at a rate which slows down and approaches zero in the infinite future. The model is known as the Einstein - de Sitter model.

The magenta curve represents a universe which contains enough matter to allow gravity to stop and reverse its expansion. As a result, it eventually collapses back onto itself in a Big Crunch after about 28 Gyr of existence. This closed universe has positive curvature, i.e. a spherical geometry, and would presently be 6.4 Gyr old and still in its expansion phase.

3 Age of the universe

Equation (28) allows calculating the age of the universe. Moving dx and dt each to one side of the equation and integrating over x ranging from 0 to 1 yields a value for t_0. Remember that $x(t)$ was defined such that $x(t_0) = 1$.

$$\frac{dx}{dt} = H_0 \sqrt{\frac{\Omega_{r,0}}{x^2} + \frac{\Omega_{m,0}}{x} + \Omega_{\Lambda,0}\, x^2 + 1 - \Omega_{r,0} - \Omega_{m,0} - \Omega_{\Lambda,0}}$$

$$dt = \frac{dx}{H_0 \sqrt{\frac{\Omega_{r,0}}{x^2} + \frac{\Omega_{m,0}}{x} + \Omega_{\Lambda,0}\, x^2 + 1 - \Omega_{r,0} - \Omega_{m,0} - \Omega_{\Lambda,0}}}$$

$$t_0 = \int_0^1 \frac{dx}{H_0 \sqrt{\frac{\Omega_{r,0}}{x^2} + \frac{\Omega_{m,0}}{x} + \Omega_{\Lambda,0}\, x^2 + 1 - \Omega_{r,0} - \Omega_{m,0} - \Omega_{\Lambda,0}}}$$

The age of the universe t_0 is shown as a function of the Hubble constant H_0 for different combinations of density parameters $\Omega_{i,0}$ in figure 3.

The highest curve for which $\Omega_{\Lambda,0} = 0$, gives a value for t_0 of about 13 billion years for values of H_0 such as those obtained in observations with the Planck and WMAP satellites. An age of 13 billion years is still less than the age of the oldest known object in the universe [12]. This shows that models without cosmological constant are not very well matching current observations.

The re-introduction of the cosmological constant in Einstein's equations provides a way to bridge the gap between theory and observations. Observational data from the Planck and WMAP satellites neighbour the curve for which $\Omega_{m,0} \approx 0.3$ and $\Omega_{\Lambda,0} \approx 0.7$. This curve represents the model which is widely accepted as best fitting the observations. The model is known as the ΛCDM model, shorthand for Λ Cold Dark Matter, referring to its main constituents.

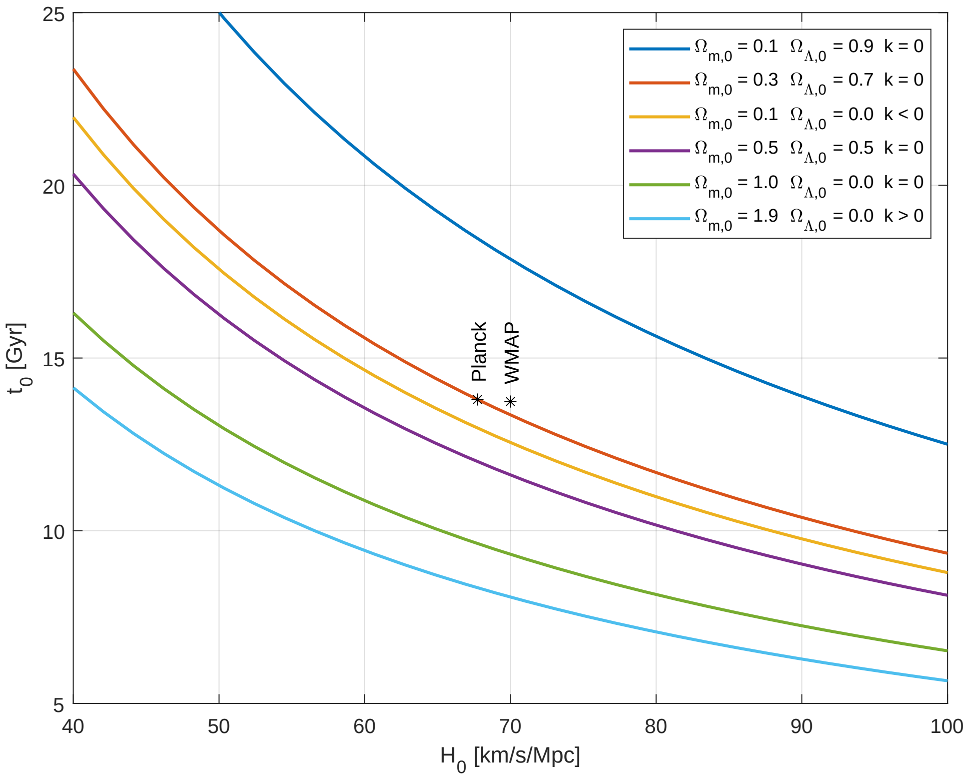

Figure 3: The age of the universe t_0 as a function of the Hubble constant H_0 for different models with no radiation contribution ($\Omega_{r,0} = 0$).

The parameters of the ΛCDM model[13] are summarized in table 1.

[12]The oldest and most distant known galaxy GN-z11 is observed in the constellation Ursa Major as it existed 13.4 billion years ago.

[13]Barbara Ryden, *Introduction to Cosmology*, 2nd Edition, page 96

Ingredient	Ω_0
photons	$\Omega_{\gamma,0} = 5.35 \times 10^{-5}$
neutrinos	$\Omega_{\nu,0} = 3.65 \times 10^{-5}$
total radiation	$\Omega_{r,0} = 9.00 \times 10^{-5}$
baryonic matter	$\Omega_{b,0} = 0.048$
dark matter	$\Omega_{d,0} = 0.262$
total matter	$\Omega_{m,0} = 0.31$
cosmological constant	$\Omega_{\Lambda,0} \approx 0.69$

Table 1: Parameters of the ΛCDM model.

This leads to a relative distribution of the components of the universe as shown in figure 4. The term dark energy refers to the cosmological constant, giving it a physical meaning while at the same time expressing it is not yet well understood. It is noteworthy that about 95% of the content of the universe is of unknown origin!

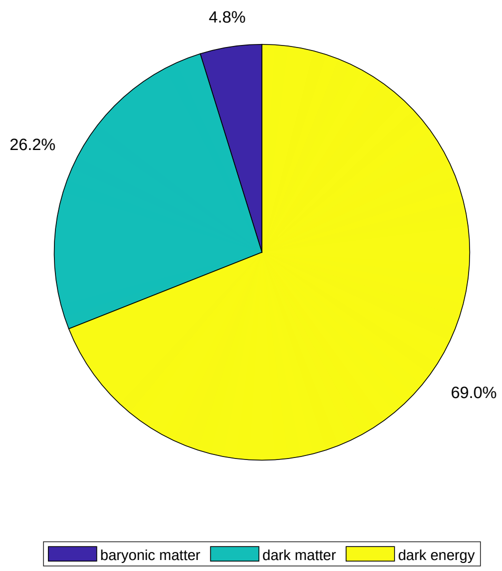

Figure 4: Relative abundance of the different components of the universe according the benchmark model.

A Pressure of a particle gas

The momentum $\vec{p}$ of a particle with energy E moving at a velocity $\vec{v}$ is given by:

$$\vec{p} = \frac{E}{c^2}\,\vec{v}$$

This is valid both for massive particles moving at non-relativistic speeds as for massless particles moving at speeds close the the speed of light.

Decomposing the vectors in 3 perpendicular components yields in x-direction:

$$p_x = \frac{E}{c^2}\,v_x$$

Similar equations apply for p_y and p_z.

Consider a cubical container with side L and volume V, filled with a gas consisting of N particles randomly moving around at an average speed $\bar{v}$. When a particle elastically collides with a wall of the container perpendicular to the x-direction, its momentum before and after the collision is the same but with opposite sign. Its change in momentum Δp is therefore:

$$\Delta p = p_x - (-p_x) = 2\,p_x$$

Using the expression obtained earlier for the momentum p_x in x-direction, the change in momentum becomes:

$$\Delta p = 2\,\frac{E}{c^2}\,v_x$$

On average, the particle collides with the wall of the container at intervals Δt given by:

$$\Delta t = \frac{2\,L}{v_x}$$

The force exerted on the wall of the container by that particle is:

$$F_i = \frac{\Delta p}{\Delta t} = \frac{E}{L}\,\frac{v_x^2}{c^2}$$

Summarizing over all the particles in the container, the total force exerted is:

$$F = \sum_{i=1}^{N} F_i = \frac{N\,E}{L}\,\frac{\bar{v}_x^2}{c^2}$$

As the particles are randomly moving around, there is no preferred direction and statistically:

$$\bar{v}_x^2 = \bar{v}_y^2 = \bar{v}_z^2$$

Using this equality in Pythagoras' theorem $\bar{v}^2 = \bar{v}_x^2 + \bar{v}_y^2 + \bar{v}_z^2$ yields:

$$\bar{v}_x^2 = \frac{1}{3}\,\bar{v}^2$$

The pressure exerted by the particles in the container is the force per unit of area or:

$$P = \frac{F}{L^2} = \frac{1}{3}\,\frac{N\,E}{L^3}\,\frac{\bar{v}^2}{c^2}$$

In terms of the energy density ϵ this becomes:

$$P = \frac{1}{3}\,\frac{\bar{v}^2}{c^2}\,\epsilon$$

In other words, the pressure P exerted by a gas of particles is proportional to its energy density ϵ with a proportionality factor w given by:

$$w = \frac{1}{3}\,\frac{\bar{v}^2}{c^2}$$