Numerical Models for Stellar Structure

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Abstract

A simplified stellar model is built using the fundamental laws of stellar structure. The model is then applied to the Sun, making use of numerical software to solve the system of differential equations, and results compared with Standard Solar Model BS05(AGS,OP) [BSB05]. The validity of the simplified model for stars with masses inferior and superior to the mass of the Sun, is also briefly investigated.

The model source code, written in the m-language and suitable for both MATLAB[1] and GNU Octave[2], is available via the GitHub repository https://www.github.com/rlanssiers/stellarstructure.

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[1] https://www.mathworks.com
[2] https://www.gnu.org/software/octave

1 Nomenclature

1.1 Physical properties

In what follows, the symbols and units listed in table 1 are used for the relevant physical properties.

Symbol	Physical Property	Unit
ϵ	energy production rate (per unit of mass)	$\mathrm{W\ kg^{-1}}$
κ	opacity	$\mathrm{m^2\ kg^{-1}}$
L	luminosity (across a spherical shell with radius r)	W
M	mass (contained in a sphere with radius r)	kg
μ	mean molecular weight	-
P	pressure	Pa
r	radius	m
ρ	mass density	$\mathrm{kg\ m^{-3}}$
T	temperature	K
V	volume	$\mathrm{m^3}$
X	hydrogen mass fraction	-
Y	helium mass fraction	-
Z	heavier elements mass fraction	-

Table 1: Symbols and units used for physical properties.

Central values are written with subscript c, e.g. P_c for the central pressure and ρ_c for the central mass density of the star. Surface values get a subscript s, e.g. T_s for the surface temperature and L_s for the surface luminosity.

1.2 Conversion factors

Astrophysicists generally continue to use the CGS system of units while this document adheres to the International System of Units (SI units). To relate formulae based on the CGS system to formulae in SI units, the conversion factors listed in table 2 have to be taken into account.

CGS Unit		SI Unit
$1\ \mathrm{g\ cm^{-3}}$	=	$10^{-3}\ \mathrm{kg\ m^{-3}}$
$1\ \mathrm{dyn}$	=	$10^{-5}\ \mathrm{N}$
$1\ \mathrm{dyn\ cm^{-2}}$	=	$10^{-1}\ \mathrm{Pa}$
$1\ \mathrm{erg}$	=	$10^{-7}\ \mathrm{J}$
$1\ \mathrm{erg\ g^{-1}\ s^{-1}}$	=	$10^{-4}\ \mathrm{W\ kg^{-1}}$
$1\ \mathrm{cm^2\ g^{-1}}$	=	$10^{-1}\ \mathrm{m^2\ kg^{-1}}$

Table 2: Conversion factors between CGS and SI units.

1.3 Physical constants

In what follows, the symbols and values of table 3 are used for the relevant physical constants [CO17, Appendix A]. All values are expressed in SI units.

Symbol	Physical Constant	Value
a	radiation constant	7.565767×10^{-16} J m^{-3} K^{-4}
c	speed of light in vacuum	2.99792458×10^{8} m s^{-1}
G	gravitational constant	6.67428×10^{-11} N m^{2} kg^{-2}
k_B	Boltzmann's constant	$1.3806504 \times 10^{-23}$ J K^{-1}
m_H	hydrogen mass	$1.673532499 \times 10^{-27}$ kg
σ	Stefan-Boltzmann constant	5.670400×10^{-8} W m^{-2} K^{-4}

Table 3: Physical constants.

1.4 Astronomical constants

It is common practise to relate the mass, luminosity and radius of a star to the Sun's mass, luminosity and radius. For that purpose, the values of the astronomical constants in table 4 are used [CO17, Appendix A].

Symbol	Astronomical Constant	Value
$M_\odot$	Solar mass	1.9891×10^{30} kg
$L_\odot$	Solar luminosity	3.839×10^{26} W
$R_\odot$	Solar radius	6.95508×10^{8} m

Table 4: Astronomical constants.

2 Differential equations of stellar structure

2.1 Introduction

The differential equations derived in the following sections constitute a simplified model of a star in the sense that the equations assume a spherically symmetrical star. This means that the star's properties spatially only depend on a single parameter, namely the distance r from the center.

2.2 Conservation of mass

Consider a spherical shell with thickness dr located at a distance r from the center of a star as shown in figure 1.

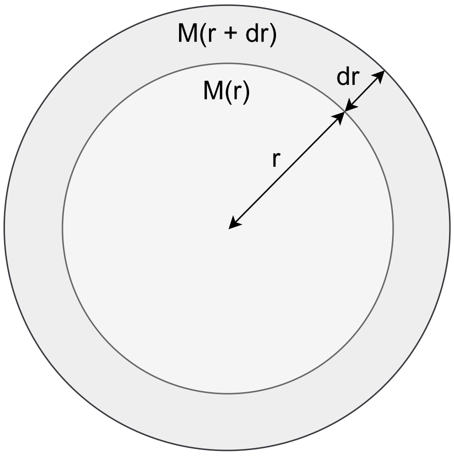

Figure 1: Spherical shell with thickness dr located at a distance r from the center of a star.

If $M(r)$ is the mass contained in the sphere with radius r, then the mass of the shell is $M(r+dr)-M(r)$, which must be equal to the shell's mass density ρ multiplied by its volume dV:

$$M(r + dr) - M(r) = \rho\, dV$$

The volume of the shell is given by:

$$dV = 4\,\pi\, r^2\, dr$$

Combining both expressions yields:

$$M(r + dr) - M(r) = \rho\, 4\,\pi\, r^2\, dr$$

$$\frac{M(r + dr) - M(r)}{dr} = \rho\, 4\,\pi\, r^2$$

$$\boxed{\frac{dM}{dr} = \rho\, 4\,\pi\, r^2} \tag{1}$$

2.3 Hydrostatic equilibrium

Consider a spherical shell with thickness dr located at a distance r from the center of a star and in particular, an element with surface area dA in that shell as shown in figure 2.

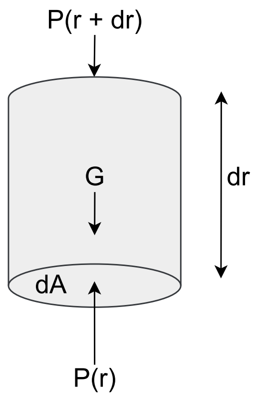

Figure 2: Element with surface area dA in a spherical shell with thickness dr located at a distance r from the center of a star.

The pressure $P(r)$ exerted on the element's inward surface creates an outward force on the element. Similarly, the pressure $P(r + dr)$ exerted on the outward surface creates an inward force. Additionally, the element with mass $\rho\, dA\, dr$ is subject to an inward gravitational force caused by the mass $M(r)$ contained in the sphere below it. If the element is in equilibrium, the sum of the 3 forces acting upon it must be zero (outward is taken as positive) and yields:

$$P(r)\, dA - P(r + dr)\, dA - \frac{G\, M\, \rho\, dA\, dr}{r^2} = 0$$

$$\frac{P(r + dr) - P(r)}{dr} = -\frac{G\, M\, \rho}{r^2}$$

$$\boxed{\frac{dP}{dr} = -\frac{G\, M\, \rho}{r^2}} \tag{2}$$

2.4 Conservation of energy

Consider a spherical shell with thickness dr located at a distance r from the center of a star as shown in figure 3.

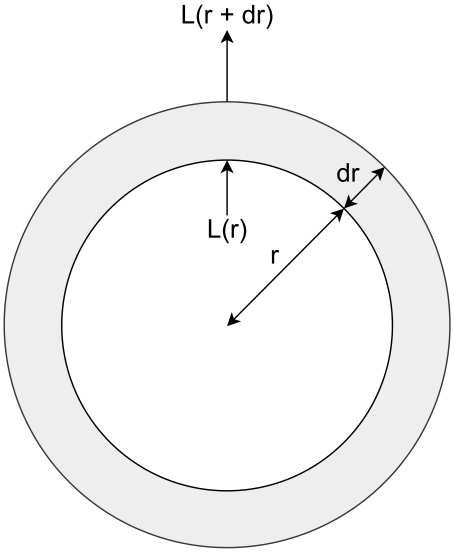

Figure 3: Spherical shell with thickness dr located at a distance r from the center of a star.

The net outward energy flow $L(r + dr) - L(r)$ must be equal to the energy generated inside the shell, which is the energy production rate ϵ multiplied by the mass of the shell $\rho\, dV$:

$$L(r + dr) - L(r) = \epsilon\, \rho\, dV$$

The volume of the shell is given by:

$$dV = 4\, \pi\, r^2\, dr$$

Combining both expressions yields:

$$L(r + dr) - L(r) = \epsilon\, \rho\, 4\, \pi\, r^2\, dr$$

$$\frac{L(r + dr) - L(r)}{dr} = \epsilon\, \rho\, 4\, \pi\, r^2$$

$$\boxed{\frac{dL}{dr} = \epsilon\, \rho\, 4\, \pi\, r^2} \tag{3}$$

2.5 Energy transport

2.5.1 Radiative transport

The flux F (the energy flow per square meter per second) in case of radiative energy transport is proportional to the temperature gradient:

$$F = -\lambda \, \frac{dT}{dr} \tag{4}$$

The proportionality factor λ is known as the coefficient of radiative conductivity. It is related to the opacity κ, i.e. the resistance to heat flow, by the following relation in which a is the radiation constant, c the speed of light, T the temperature and ρ the mass density:

$$\lambda = \frac{4\, a\, c\, T^3}{3\, \kappa\, \rho} \tag{5}$$

For a sphere with radius r, the flux F is related to the luminosity L (the energy flow per second) by:

$$F = \frac{L}{4\, \pi\, r^2} \tag{6}$$

Substituting equations (5) and (6) into equation (4) leads to:

$$\frac{L}{4\, \pi\, r^2} = -\frac{4\, a\, c\, T^3}{3\, \kappa\, \rho} \, \frac{dT}{dr}$$

$$\boxed{\frac{dT}{dr} = -\frac{3\, \kappa\, \rho\, L}{16\, \pi\, a\, c\, r^2\, T^3}} \tag{7}$$

2.5.2 Convective transport

Convective energy transport in stellar interiors is thought to occur via hot bubbles of gas rising and expanding adiabatically, i.e. without exchanging heat with their surroundings, until they finally dissolve in the surrounding gas and disappear [CO17, p. 321]. The first law of thermodynamics gives rise to the equation which relates the volume V of a gas to its pressure P while undergoing a reversible adiabatic process, i.e. a process with no entropy generation and without heat exchange:

$$P\, V^{\gamma} = K \tag{8}$$

In this adiabatic gas law, K is a constant and exponent γ the adiabatic index. The latter is defined as the ratio between the specific heat[3] at constant pressure C_P and the specific heat at constant volume C_V:

$$\gamma \equiv \frac{C_P}{C_V}$$

[3] The specific heat of a substance is the heat capacity of a sample divided by the mass of the sample or in other words, the amount of heat that needs to be added per unit of mass to achieve one unit of temperature increase.

The adiabatic index for an ideal monatomic gas[4] equals $5/3$ or about 1.67.

When expressing the volume V in terms of the mass density ρ of the mass-element M, equation (8) becomes:

$$P \left(\frac{M}{\rho}\right)^{\gamma} = K$$

$$P = \frac{K}{M^{\gamma}}\,\rho^{\gamma} \tag{9}$$

Think of an insulated piston, extended over a distance r, with internal volume V and internal pressure P. The mass M of the gas in the piston remains constant whereas the mass density ρ changes with r. Differentiating both sides of equation (9) with respect to r then yields:

$$\frac{dP}{dr} = \frac{K}{M^{\gamma}}\,\gamma\,\rho^{\gamma-1}\,\frac{d\rho}{dr}$$

And by re-substituting equation (9):

$$\frac{dP}{dr} = \gamma\,\frac{P}{\rho}\,\frac{d\rho}{dr} \tag{10}$$

The ideal gas law describes how the pressure P, the volume V and the temperature T of an ideal gas of matter particles are related to each other. In the molecular form of the equation, the factor n represents the number of molecules contained in the volume V while k_B is Boltzmann's constant:

$$P\,V = n\,k_B\,T \tag{11}$$

The particle number densities n/V of both nuclei and electrons for a fully ionized gas composed of a mass fraction X of hydrogen, a mass fraction Y of helium and a mass fraction Z of all elements heavier than helium are listed in table 5. Astronomers usually refer to all elements heavier than helium as metals and Z is therefore also known as the metallicity of a star. Obviously, the sum of X, Y and Z equals 1. For sake of simplicity, metals are considered here to have an average mass $A\,m_H$ and approximately $A/2$ electrons per nucleus.

[4] A monatomic gas is one in which atoms are not bound to each other.

Particle	Hydrogen	Helium	Metals
nuclei	$\frac{\rho\,X}{m_H}$	$\frac{\rho\,Y}{4\,m_H}$	$\frac{\rho\,Z}{A\,m_H}$
electrons	$\frac{\rho\,X}{m_H}$	$\frac{2\,\rho\,Y}{4\,m_H}$	$\frac{A}{2}\,\frac{\rho\,Z}{A\,m_H}$

Table 5: Particle number densities in a fully ionized stellar plasma based on the mass density ρ, the mass of the hydrogen atom m_H, the hydrogen mass fraction X, the helium mass fraction Y and the metals mass fraction Z. The factor A stands for the average mass of the metal atoms compared to the hydrogen atom.

The overall particle number density is then:

$$\frac{n}{V} = \underbrace{\left[2\,X + \frac{3}{4}\,Y + \left(\frac{1}{A} + \frac{1}{2}\right)Z\right]}_{1/\mu}\,\frac{\rho}{m_H} \tag{12}$$

The factor μ is the mean molecular weight which approximately equals (if $A >> 1$):

$$\boxed{\mu \approx \frac{1}{2\,X + \frac{3}{4}\,Y + \frac{1}{2}\,Z}} \tag{13}$$

In reality, the composition of a star and consequently μ is not homogeneous from center to surface (and neither does it change linearly with radius).

Using expression (12) for the particle number density, the ideal gas law (11) transforms into:

$$P = \frac{k_B\,T}{\mu\,m_H}\,\rho \tag{14}$$

Differentiating both sides of equation (14) with respect to r yields:

$$\frac{dP}{dr} = -\frac{\rho\,k_B\,T}{m_H}\,\mu^{-2}\,\frac{d\mu}{dr} + \frac{k_B\,T}{\mu\,m_H}\,\frac{d\rho}{dr} + \frac{\rho\,k_B}{\mu\,m_H}\,\frac{dT}{dr}$$

And by re-substituting equation (14):

$$\frac{dP}{dr} = -\frac{P}{\mu}\,\frac{d\mu}{dr} + \frac{P}{\rho}\,\frac{d\rho}{dr} + \frac{P}{T}\,\frac{dT}{dr}$$

Substituting equation (10) and assuming that the mean molecular weight μ is constant throughout the star and consequently $d\mu/dr = 0$ gives:

$$\frac{dP}{dr} = \frac{1}{\gamma}\frac{dP}{dr} + \frac{P}{T}\frac{dT}{dr}$$

$$\boxed{\frac{dT}{dr} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}} \tag{15}$$

2.5.3 Criterion for convection

With 2 modes of energy transport[5], the question arises when one dominates over the other? The premise is, that a region is convectively unstable when the actual temperature gradient is steeper than the adiabatic temperature gradient [Gui19]:

$$\frac{dT}{dr} < \left(\frac{dT}{dr}\right)_{ad} \tag{16}$$

Note that in a star, the temperature decreases from center to surface which means that dT/dr is negative. Hence why ”steeper” translates into ”less than”. Taking equation (15) for the adiabatic temperature gradient into account, another way to formulate criterion (16) is[6]:

$$\frac{dT}{dr} < \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}$$

$$\frac{T}{P}\frac{dP}{dr}\frac{dr}{dT} < \frac{\gamma}{\gamma - 1}$$

$$\boxed{\frac{d\ln P}{d\ln T} < \frac{\gamma}{\gamma - 1}}$$

Using equations (7) and (15) and the equation of hydrostatic equilibrium (2) to eliminate dP/dr, it is possible to define a critical luminosity L_{crit} based on expression (16). Where the actual luminosity exceeds the critical luminosity, a region is convectively unstable:

$$-\frac{3\,\kappa\,\rho\,L}{16\,\pi\,a\,c\,r^2\,T^3} < \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}$$

$$\frac{3\,\kappa\,\rho\,L}{16\,\pi\,a\,c\,r^2\,T^3} > \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{G\,M\,\rho}{r^2}$$

[5]There are actually 3 modes of energy transport but the conductive contribution is considered to be negligible inside stars.
[6]Recall that dividing by a negative quantity reverses the comparison operator.

$$\boxed{L > \underbrace{\left(1 - \frac{1}{\gamma}\right) \frac{16\,\pi\,a\,c\,G\,M\,T^4}{3\,\kappa\,P}}_{=L_{crit}}}$$

Figure 4 illustrates the different possible situations. Note that there may be successive zones inside a star dominated by either radiative or convective energy transport.

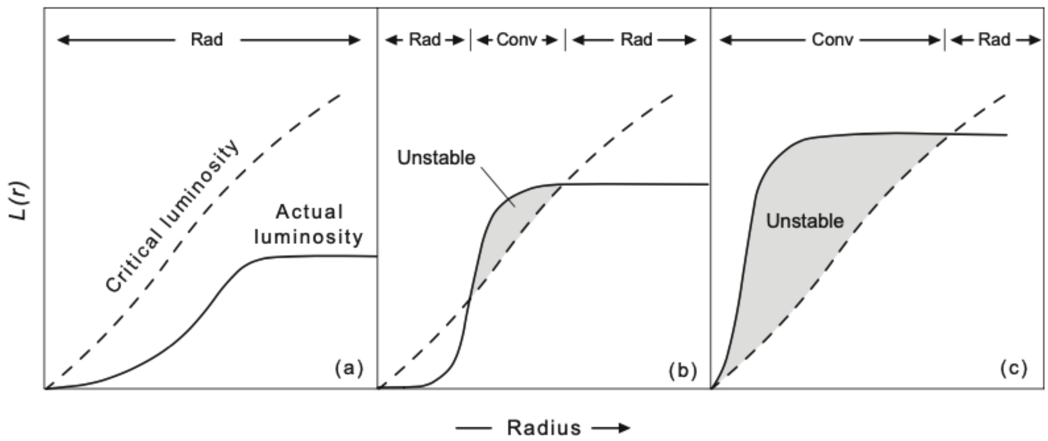

Figure 4: Actual luminosity (solid line) compared to critical luminosity (dashed line) inside a star. Where the actual luminosity exceeds the critical luminosity, a region is convectively unstable. (Credit: Mike Guidry)

3 Constitutive relations

3.1 Introduction

Equations (1), (2), (3) and (7) or (15) form a system of 4 differential equations which by themselves are not sufficient to solve for the 7 variables M, P, L, T, ρ, ϵ and κ. To be able to find a unique solution describing the internal structure of a star, they need to be supplemented by at least 3 additional equations.

3.2 Equation of state

The ideal gas law (14) which was introduced earlier, deals with the pressure exerted by a gas of matter particles:

$$P_{gas} = \frac{k_B\,T}{\mu\,m_H}\,\rho$$

Additionally, photons also exert a pressure which is proportional to the 4th power of the temperature T, with 1/3 of the radiation constant a as proportionality factor:

$$P_{rad} = \frac{1}{3}\, a\, T^4$$

The sum of the gas pressure P_{gas} and the radiation pressure P_{rad} gives the total pressure P:

$$P = P_{gas} + P_{rad}$$

$$P = \frac{k_B\, T}{\mu\, m_H}\, \rho + \frac{1}{3}\, a\, T^4$$

Rearranging the terms leads to an equation for the mass density ρ as a function of the pressure P, the temperature T and the mean molecular weight μ:

$$\boxed{\rho = \left(P - \frac{1}{3}\, a\, T^4\right) \frac{\mu\, m_H}{k_B\, T}} \tag{17}$$

3.3 Energy production rate

3.3.1 Introduction

The energy produced in stars is generated by different simultaneous nuclear fusion reactions. For stars which are still mainly burning[7] hydrogen, the so called zero age main sequence or ZAMS stars, the dominant processes are the proton-proton and the carbon-nitrogen-oxygen reactions.

3.3.2 Proton-proton chains

In the proton-proton reactions, hydrogen $^1\mathrm{H}$ fuses to helium $^4\mathrm{He}$ via intermediate hydrogen and helium isotopes. There a 2 side branches to the reaction which also yield $^4\mathrm{He}$ via yet other intermediate isotopes like $^7\mathrm{Be}$, $^7\mathrm{Li}$, $^8\mathrm{B}$ and $^8\mathrm{Be}$ as shown below:

$$\text{PP-I} \quad \begin{cases} ^1\mathrm{H} + {}^1\mathrm{H} & \to {}^2\mathrm{H} + e^+ + \nu_e \\ ^2\mathrm{H} + {}^1\mathrm{H} & \to {}^3\mathrm{He} + \gamma \\ ^3\mathrm{He} + {}^3\mathrm{He} & \to {}^4\mathrm{He} + {}^1\mathrm{H} + {}^1\mathrm{H} \end{cases}$$

$$\text{PP-II} \quad \begin{cases} ^3\mathrm{He} + {}^4\mathrm{He} & \to {}^7\mathrm{Be} + \gamma \\ ^7\mathrm{Be} + e^- & \to {}^7\mathrm{Li} + \nu_e + \gamma \\ ^7\mathrm{Li} + {}^1\mathrm{H} & \to {}^4\mathrm{He} + {}^4\mathrm{He} \end{cases}$$

$$\text{PP-III} \quad \begin{cases} ^7\mathrm{Be} + {}^1\mathrm{H} & \to {}^8\mathrm{B} + \gamma \\ ^8\mathrm{B} & \to {}^8\mathrm{Be} + e^+ + \nu_e \\ ^8\mathrm{Be} & \to {}^4\mathrm{He} + {}^4\mathrm{He} \end{cases}$$

[7] Stars are fueled by nuclear fusion reactions which are of a completely different nature than the chemical combustion reactions (of hydrocarbons with oxygen) used on Earth as energy source.

An approximation for the proton-proton energy production rate ϵ_{pp} as a function of the mass density ρ, the hydrogen mass fraction X and the temperature T, with T_6 being a short notation for $T/10^6$, is given by [CO17, p. 311]:

$$\boxed{\epsilon_{pp} \approx 0.241\, \rho\, X^2\, f_{pp}\, \psi_{pp}\, C_{pp}\, T_6^{-2/3}\, e^{-33.80\, T_6^{-1/3}}}$$

Factor f_{pp} is the proton-proton chain screening factor, ψ_{pp} a correction factor that accounts for the simultaneous occurrence of the PP-I, PP-II and PP-III chains and C_{pp} a higher correction term. Without making a too large error, for practical purposes:

$$\boxed{\begin{aligned} f_{pp} &\approx 1 \\ \psi_{pp} &\approx 1 \\ C_{pp} &\approx 1 \end{aligned}}$$

3.3.3 Carbon-nitrogen-oxygen cycles

In the CNO reactions, hydrogen $^1\mathrm{H}$ fuses to helium $^4\mathrm{He}$ using carbon, nitrogen and oxygen isotopes as catalysts. The reaction actually consists of 3 subbranches as shown below:

$$\text{CNO-I} \begin{cases} ^{12}\mathrm{C} + {}^1\mathrm{H} & \rightarrow {}^{13}\mathrm{N} + \gamma \\ ^{13}\mathrm{N} & \rightarrow {}^{13}\mathrm{C} + e^+ + \nu_e \\ ^{13}\mathrm{C} + {}^1\mathrm{H} & \rightarrow {}^{14}\mathrm{N} + \gamma \\ ^{14}\mathrm{N} + {}^1\mathrm{H} & \rightarrow {}^{15}\mathrm{O} + \gamma \\ ^{15}\mathrm{O} & \rightarrow {}^{15}\mathrm{N} + e^+ + \nu_e \\ ^{15}\mathrm{N} + {}^1\mathrm{H} & \rightarrow {}^{12}\mathrm{C} + {}^4\mathrm{He} \end{cases}$$

$$\text{CNO-II} \begin{cases} ^{15}\mathrm{N} + {}^1\mathrm{H} & \rightarrow {}^{16}\mathrm{O} + \gamma \\ ^{16}\mathrm{O} + {}^1\mathrm{H} & \rightarrow {}^{17}\mathrm{F} + \gamma \\ ^{17}\mathrm{F} & \rightarrow {}^{17}\mathrm{O} + e^+ + \nu_e \\ ^{17}\mathrm{O} + {}^1\mathrm{H} & \rightarrow {}^{14}\mathrm{N} + {}^4\mathrm{He} \\ ^{14}\mathrm{N} + {}^1\mathrm{H} & \rightarrow {}^{15}\mathrm{O} + \gamma \\ ^{15}\mathrm{O} & \rightarrow {}^{15}\mathrm{N} + e^+ + \nu_e \end{cases}$$

$$\text{CNO-III} \begin{cases} {}^{17}\mathrm{O} + {}^{1}\mathrm{H} & \rightarrow {}^{18}\mathrm{F} + \gamma \\ {}^{18}\mathrm{F} + e^{-} & \rightarrow {}^{18}\mathrm{O} + \nu_e \\ {}^{18}\mathrm{O} + {}^{1}\mathrm{H} & \rightarrow {}^{19}\mathrm{F} + \gamma \\ {}^{19}\mathrm{F} + {}^{1}\mathrm{H} & \rightarrow {}^{16}\mathrm{O} + {}^{4}\mathrm{He} \\ {}^{16}\mathrm{O} + {}^{1}\mathrm{H} & \rightarrow {}^{17}\mathrm{F} + \gamma \\ {}^{17}\mathrm{F} & \rightarrow {}^{17}\mathrm{O} + e^{+} + \nu_e \end{cases}$$

Essentially, helium, positrons, electron neutrinos and photons are being produced via the following endless transformation cycles:

$$\text{CNO-I : } {}^{12}\mathrm{C} \rightarrow {}^{13}\mathrm{N} \rightarrow {}^{13}\mathrm{C} \rightarrow {}^{14}\mathrm{N} \rightarrow {}^{15}\mathrm{O} \rightarrow {}^{15}\mathrm{N} \rightarrow {}^{12}\mathrm{C}$$

$$\text{CNO-II : } {}^{15}\mathrm{N} \rightarrow {}^{16}\mathrm{O} \rightarrow {}^{17}\mathrm{F} \rightarrow {}^{17}\mathrm{O} \rightarrow {}^{14}\mathrm{N} \rightarrow {}^{15}\mathrm{O} \rightarrow {}^{15}\mathrm{N}$$

$$\text{CNO-III : } {}^{17}\mathrm{O} \rightarrow {}^{18}\mathrm{F} \rightarrow {}^{18}\mathrm{O} \rightarrow {}^{19}\mathrm{F} \rightarrow {}^{16}\mathrm{O} \rightarrow {}^{17}\mathrm{F} \rightarrow {}^{17}\mathrm{O}$$

In each cycle, 4 hydrogen ${}^{1}\mathrm{H}$ nuclei are ”consumed” to form 1 helium ${}^{4}\mathrm{He}$ nucleus.

An approximation for the CNO cycle energy production rate ϵ_{CNO} as a function of the mass density ρ, the hydrogen mass fraction X, the metallicity Z and the temperature T, with T_6 being a short notation for $T/10^6$, is given by [CO17, p. 312]:

$$\boxed{\epsilon_{CNO} \approx 8.67 \times 10^{20} \, \rho \, X \, X_{CNO} \, C_{CNO} \, T_6^{-2/3} \, e^{-152.28 \, T_6^{-1/3}}}$$

X_{CNO} is the total mass fraction of carbon, nitrogen and oxygen and C_{CNO} a higher order correction term. Without making a too large error, for practical purposes:

$$\boxed{\begin{aligned} X_{CNO} &\approx \frac{Z}{2} \\ C_{CNO} &\approx 1 \end{aligned}}$$

3.3.4 Total energy production rate

The total energy production rate is the sum of the energy production rate from the proton-proton chains and the carbo-nitrogen-oxygen cycles:

$$\boxed{\epsilon = \epsilon_{pp} + \epsilon_{CNO}} \tag{18}$$

Depending on the temperature, either the first or the second term of equation (18) is dominant. Figure 5 plots the energy production rate per mass density and per (main) mass fraction $\epsilon_{pp}/\rho X^2$ and $\epsilon_{CNO}/\rho XZ$ for a range of temperatures. It matches similar figures available in the literature [HKT04, p. 306] and shows that the proton-proton chains dominate the energy production up to a temperature of about 14×10^6 K. At higher temperatures, the CNO cycles rapidly take over.

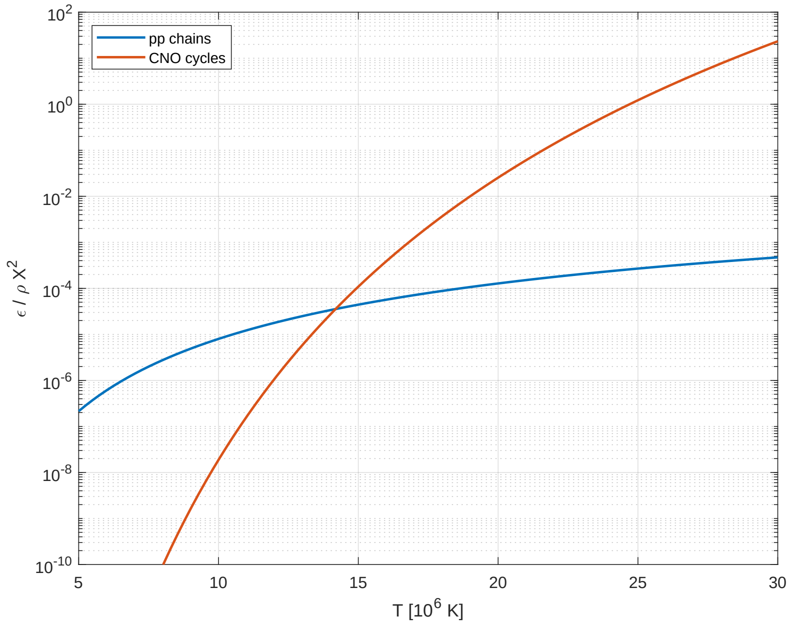

Figure 5: Energy production rate per mass density and per (main) mass fraction as a function of temperature for the proton-proton chains and the CNO cycles.

3.4 Opacity

3.4.1 Introduction

The photons produced by the nuclear reactions inside a star cannot freely escape to the star's surface due to their interactions with other particles. As such, the stellar interior is not transparent for photons but instead, has a certain degree of opacity which depends on several factors. The 4 main types of interaction which are responsible for the opacity are:

Electron scattering is the process in which a photon is scattered around when it encounters a free electron on its path.

Free-free absorption involves a free electron and an ion in close proximity so that electromagnetic coupling between the 2 can serve as a bridge to transfer photon energy and momentum. A free electron cannot absorb a photon because in that case, conservation of energy and momentum cannot both be satisfied.

Bound-free absorption (ionization) occurs when photon energy is used to free an electron from the nucleus to which it is bound.

Bound-bound absorption (excitation) occurs when photon energy is used to increase the energy level of an electron bound to a nucleus.

Due to the complexity of the involved processes, opacities for professional stellar models are available in tabular form for many stellar mixtures over wide ranges of mass density and temperature. Simplified models generally use approximate equations for the different contributions to the total opacity, all of which have the form:

$$\kappa \approx \kappa_0 \, \rho^{\alpha} \, T^{\beta} \tag{19}$$

In cases where $\alpha = 1$ and $\beta = -3.5$, this relationship is known as Kramers' law, named after the Dutch physicist Hendrik Kramers who worked, among other subjects, on the interaction between electromagnetic waves and matter.

3.4.2 Electron scattering

The approximate opacity κ_e, caused by electron scattering, is a function of the hydrogen mass fraction X and is given by [CO17, p. 250]:

$$\boxed{\kappa_{es} \approx 0.02\,(1+X)}$$

3.4.3 Free-free absorption

The approximate opacity κ_{ff}, caused by free-free absorption, is a function of the hydrogen mass fraction X, the metallicity Z, the mass density ρ and the temperature T and is given by [CO17, p. 249]:

$$\boxed{\kappa_{ff} \approx 3.68 \times 10^{18}\, g_{ff}\,(1-Z)\,(1+X)\,\rho\, T^{-3.5}}$$

Gaunt factor g_{ff} is a quantum-mechanical correction term with, for practical purposes:

$$\boxed{g_{ff} \approx 1}$$

3.4.4 Bound-free absorption

The approximate opacity κ_{bf}, caused by bound-free absorption, is a function of the hydrogen mass fraction X, the metallicity Z, the mass density ρ and the temperature T and is given by [CO17, p. 249]:

$$\boxed{\kappa_{bf} \approx 4.34 \times 10^{21}\, \frac{g_{bf}}{t}\, Z\,(1+X)\,\rho\, T^{-3.5}}$$

Again, gaunt factor g_{bf} is a quantum-mechanical correction term with, for practical purposes:

$$\boxed{g_{bf} \approx 1}$$

The guillotine factor t describes the cutoff of an atom's contribution to the opacity after it has been ionized. Typical values lie between 1 and 100.

A special case of bound-free absorption involves the negative hydrogen ion. A (neutral) hydrogen atom is indeed capable of capturing an additional electron, resulting in a negative hydrogen ion or anion which is an important constituent of stars. The bound-free interaction with a photon sets the additional electron free again from its hydrogen atom. This gives rise to an additional and specific opacity contribution. The approximate opacity κ_{H^-} is a function of the metallicity Z, the mass density ρ and the temperature T and is given by [CO17, p. 250]:

$$\boxed{\kappa_{H^-} \approx 7.9 \times 10^{-34}\, \frac{Z}{0.02}\, \sqrt{\rho}\, T^9 \quad \text{if} \quad \begin{cases} 0.001 \leqslant Z \leqslant 0.03 \\ 10^{-7}\ \mathrm{kg\ m^{-3}} \leqslant \rho \leqslant 10^{-2}\ \mathrm{kg\ m^{-3}} \\ 3000\ \mathrm{K} \leqslant T \leqslant 6000\ \mathrm{K} \end{cases}}$$

The opacity contribution due to the negative hydrogen ion is only significant within certain intervals of the metallicity, mass density and temperature as indicated. It is further not taken into account for the simplified model being developed here.

3.4.5 Bound-bound absorption

The opacity caused by bound-bound absorption is an order of magnitude smaller than the free-free and bound-free opacities and is therefore not taken into account.

3.4.6 Total opacity

The different opacity contributions are approximated by the power law of equation (19) using the values listed in table 6 for κ_0, α and β:

Opacity	κ_0	α	β
κ_{es}	$0.02\,(1+X)$	0	0
κ_{ff}	$3.68 \times 10^{18}\, g_{ff}\,(1-Z)\,(1+X)$	1	-3.5
κ_{bf}	$4.34 \times 10^{21}\, \frac{g_{bf}}{t}\, Z\,(1+X)$	1	-3.5
κ_{H^-}	$7.9 \times 10^{-34}\, \frac{Z}{0.02}$	0.5	9

Table 6: Coefficients for opacity power law $\kappa \approx \kappa_0\, \rho^\alpha\, T^\beta$.

To determine the total opacity, the contributions from the different types of interaction are to be added:

$$\boxed{\kappa = \kappa_{es} + \kappa_{ff} + \kappa_{bf} + \kappa_{H^-}} \tag{20}$$

Figure 6 plots equation (20) for different mass densities over a broad range of temperatures and broadly matches similar figures in the literature [CO17, p. 251] [HKT04, p. 217].

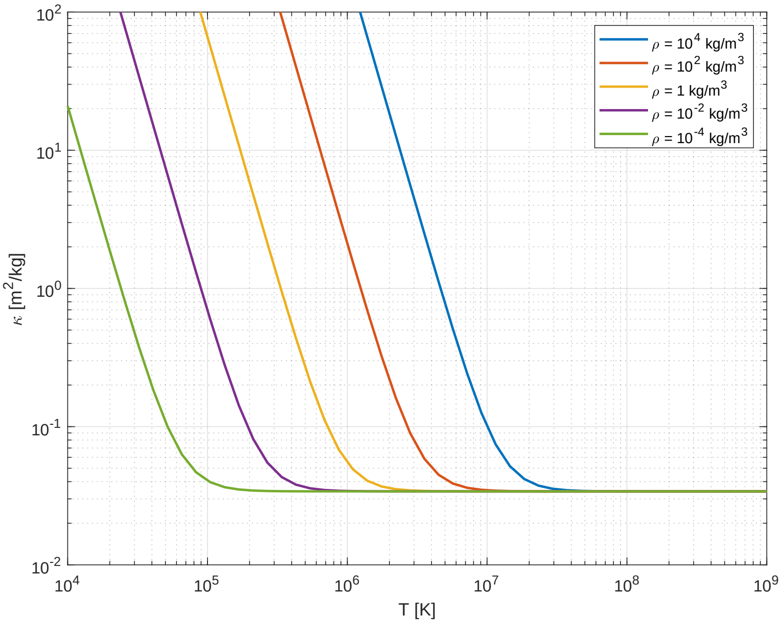

Figure 6: Total opacity as a function of temperature for a mixture with hydrogen mass fraction X = 0.70 and metallicity Z = 0.02. The distinct curves represent different mass densities.

4 Model parameters

4.1 Integration interval

Differential equations (1), (2), (3) and (7) or (15) all have the distance from the center r as independent variable. Any numerical solution valid for the entirety of the star will have to be computed with r ranging from (practically) zero up to the star's surface radius R_s. Therefore, R_s needs to be determined first.

In respectively 1905 and 1913, Ejnar Hertzsprung and Henry Norris Russell independently discovered that stars plotted in a xy-diagram with spectral type on the x-axis and absolute magnitude on the y-axis, all concentrate in certain bands across the diagram. The main feature of the Hertzsprung-Russell diagram is a narrow band reaching from the upper left to the lower right, commonly known as the main sequence.

Given that the spectral type of a star is related to its effective temperature[8] and the absolute magnitude to its luminosity, alternative (more theoretical) representations like figure 7 offer an equivalent insight.

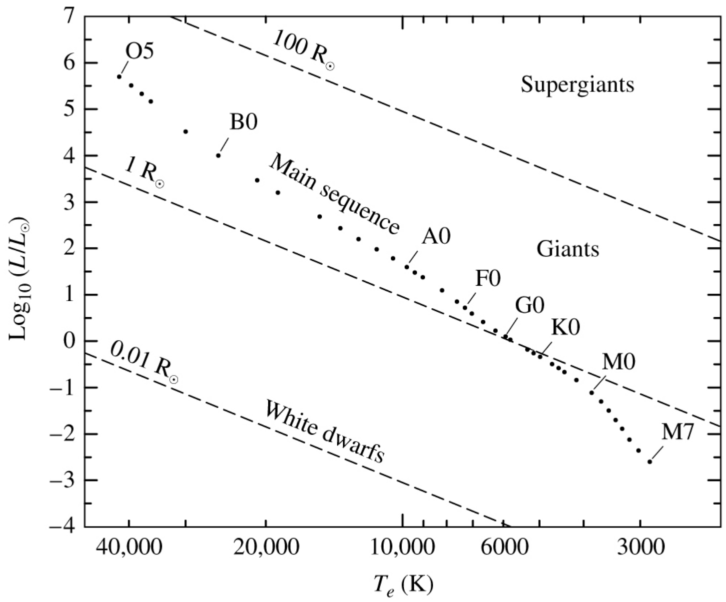

Figure 7: The theorist's Hertzsprung-Russell diagram. The dashed lines indicate lines of constant radius. (Credit: Bradley Carroll, Dale Ostlie)

Stars emit blackbody radiation and their flux F (the total amount of energy emitted per second from each square meter of surface) as a function of effective temperature T_e is therefore given by the Stefan-Boltzmann law in which σ is the Stefan-Boltzmann constant:

$$F = \sigma\, T_e^4$$

The surface area A_s of a star with surface radius R_s equals $4\,\pi\,R_s^2$ and its surface luminosity L_s (the total amount of energy emitted per second) is consequently:

$$L_s = F\, A_s$$

$$L_s = \sigma\, T_e^4\, 4\,\pi\, R_s^2$$

Reworking this equation yields the star's surface radius as:

[8] The effective temperature of a star is the temperature that a blackbody radiator needs to have to emit the same total amount of radiation.

$$\boxed{R_s = \sqrt{\frac{L_s}{4\,\pi\,\sigma\,T_e^4}}} \tag{21}$$

Equation (21) explains the lines of equal radius in the logarithmic diagram of figure 7.

Further research using sophisticated stellar models has shown that the observable properties of a main sequence star, such as its effective temperature and luminosity, are all dictated by the mass of the star. In other words, the main sequence is a mass sequence as illustrated in figure 8.

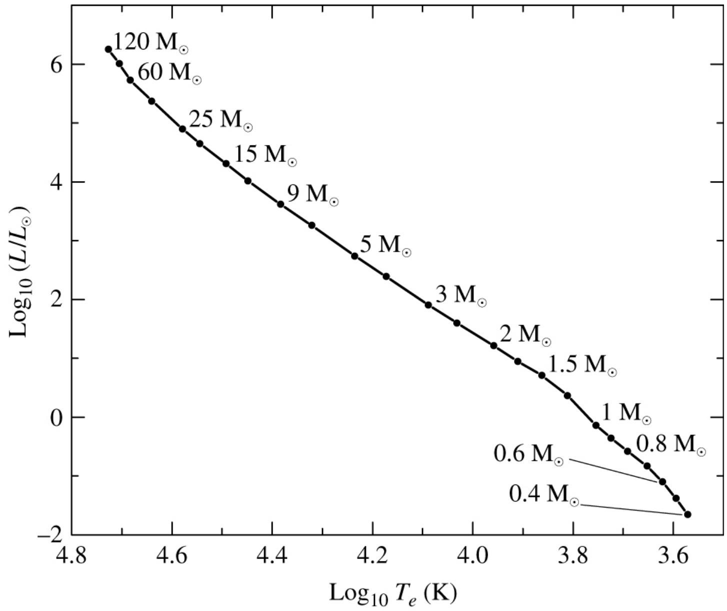

Figure 8: The locations of stellar models on a theoretical Hertzsprung-Russell diagram. (Credit: Bradley Carroll, Dale Ostlie)

Consequently, when either the effective temperature, the luminosity or the mass of a main sequence star is known, the other 2 properties are readily obtainable from the Hertzsprung-Russell diagram. That mass is a determining factor is formalized in the Vogt-Russell theorem:

> The mass and the composition structure throughout a star uniquely determine its radius, luminosity and internal structure, as well as its subsequent evolution.

4.2 Boundary conditions

Solving differential equations (1), (2), (3) and (7) or (15) requires boundary values for M, P, L and T. More notably, if the system is solved from the center to the surface, M_c, P_c, L_c and T_c have to be determined one way or another. For two of them, this is quite straightforward:

$$\left.\begin{aligned} M_c &\to 0 \\ L_c &\to 0 \end{aligned}\right\} \quad \text{as} \quad r \to 0$$

Estimating a star's central pressure and temperature is more challenging.

Solving the system of differential equations backward starting from the surface, requires the boundary values M_s, P_s, L_s and T_s. For the purpose of the simplified model, it is adequate to use zero boundary conditions for two of them:

$$\left.\begin{aligned} T_s &\to 0 \\ P_s &\to 0 \end{aligned}\right\} \quad \text{as} \quad r \to R_s$$

The result is a set of boundary values, some valid at the center and some at the surface. Professional models often employ a technique solving the model both in forward and backward direction, with reasonable estimates for the unknown boundary values. In an iterative process, the estimates are refined until both solutions transition into one another without discontinuities where they meet.

For the simplified model being developed here, the option was chosen to solve it from surface to center, with known values for M_s, P_s, L_s and T_s and to verify if the model converges and produces results in broad agreement with professional models. Unfortunately, if the surface pressure is assumed to be zero, the surface mass density is negative and meaningless by virtue of equation of state (17). If the surface pressure is assumed to be only consisting of photon pressure, the same equation of state understandably results in a zero surface mass density. But then, as the mass density appears in differential equations (1), (2), (3) and (7), the respective gradients dM/dr, dP/dr, etc. are also zero, resulting in constant and meaningless solutions. A reasonable estimate for P_s which does not imply a zero surface mass density is therefore necessary.

Dividing equation (2) by equation (7) applied to the surface of the star yields:

$$\frac{dP}{dT} = \frac{16\,\pi\,a\,c\,G\,M_s}{3\,\kappa\,L_s}\,T^3 \tag{22}$$

In the outer atmosphere of stars, there are few free electrons and the electron scattering and negative hydrogen ion contributions to the total opacity are negligible compared to the free-free and bound-free absorption contributions [CO17, p. A-24]. The total opacity is then, with $\kappa_{0,ff}$ and $\kappa_{0,bf}$ taking the values as given in table 6:

$$\kappa = (\kappa_{0,ff} + \kappa_{0,bf})\,\rho\,T^{-3.5} \tag{23}$$

Substituting equation (23) in equation (22) results in:

$$\frac{dP}{dT} = \frac{16\,\pi\,a\,c\,G\,M_s}{3\,(\kappa_{0,ff} + \kappa_{0,bf})\,\rho\,L_s}\,T^{6.5}$$

Using ideal gas law (14) to eliminate ρ gives:

$$\frac{dP}{dT} = \frac{16\,\pi\,a\,c\,G\,M_s\,k_B}{3\,(\kappa_{0,ff} + \kappa_{0,bf})\,\mu\,m_H\,L_s}\,\frac{T^{7.5}}{P}$$

$$P\,dP = \frac{16\,\pi\,a\,c\,G\,M_s\,k_B}{3\,(\kappa_{0,ff} + \kappa_{0,bf})\,\mu\,m_H\,L_s}\,T^{7.5}\,dT$$

$$d\left(\frac{1}{2}\,P^2\right) = d\left(\frac{16\,\pi\,a\,c\,G\,M_s\,k_B}{3\,(\kappa_{0,ff} + \kappa_{0,bf})\,\mu\,m_H\,L_s}\,\frac{1}{8.5}\,T^{8.5}\right)$$

$$\boxed{P = \sqrt{\frac{16\,\pi\,a\,c\,G\,M_s\,k_B}{12.75\,(\kappa_{0,ff} + \kappa_{0,bf})\,\mu\,m_H\,L_s}}\,T^{4.25}} \tag{24}$$

Equation (24) allows calculating an estimate for the surface pressure P_s based on the surface temperature T_s. A reasonable approximation for the surface temperature is given by the star's effective temperature T_e.

4.3 Free parameters

The present simplified stellar model is not a dynamic model, meaning that the results do not depend on time. More specifically, the model does not include equations describing how the internal composition of the star evolves with time. And neither does it assume a composition which varies with the distance from the center. In other words, the respective mass fractions are considered to be constants and a good estimation needs to be made for X, Y and Z, satisfying the condition:

$$X + Y + Z = 1$$

5 Results

The obvious candidate to test the simplified stellar model is the Sun. Not only is it a zero age main sequence star, there are also mathematical models available made by professional astrophysicists to compare results with.

Figure 9 shows how the mass, mass density, pressure, temperature and luminosity vary according the simplified model as a function of the relative distance from the center of the Sun. Table 7 summarizes the boundary values and other parameters which were used. The solver is programmed to stop when it reaches a negative value for any of the physical properties M, P, L, T being computed. To validate the results, the comparison is made with data[9] from Standard Solar Model BS05(AGS,OP) [BSB05].

Parameter	Value
M_s	$1\ \mathrm{M}_\odot$
P_s	from equation (24)
L_s	$1\ \mathrm{L}_\odot$
T_s	$5777\ \mathrm{K}$
X	0.70
Z	0.03
γ	5/3
R_s	from equation (21)

Table 7: Model parameters.

The solution of the simplified model fairly matches the reference model but the discrepancy between the two gets larger when the center of the star is approached. Another limitation is that the simplified model does not seem to be able to identify the convective zones in the Sun as the actual luminosity never exceeds the critical luminosity. In other words, the solution is purely radiative while it is known that above $71\ \%$ of its surface radius, the Solar interior is convective [CO17, p. 354].

[9] http://www.sns.ias.edu/ jnb/SNdata/sndata.html

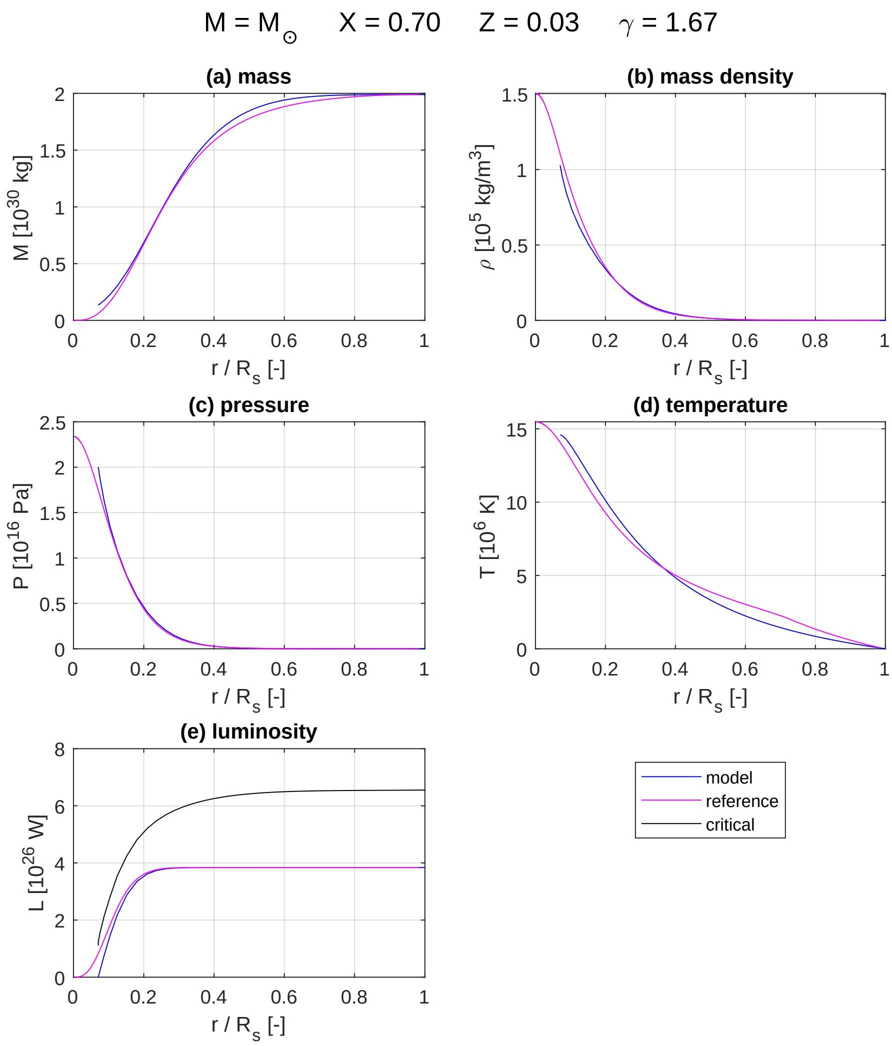

Figure 9: Solution of the simplified model applied to the Sun. The magenta reference curves represent Standard Solar Model BS05(AGS,OP).

When applied to main sequence stars with different masses, the simplified model converges to an acceptable solution albeit that in many cases, the solver halts on mass and/or luminosity reaching zero before the center of the star is reached. Figure 10 shows the result for a $17.5\ \mathrm{M}_\odot$ star and figure 11 for a $0.51\ \mathrm{M}_\odot$ star with boundary value data taken from [CO17, Appendix G]. The plots for the $17.5\ \mathrm{M}_\odot$ star clearly appear to be more realistic than those for the $0.51\ \mathrm{M}_\odot$ star.

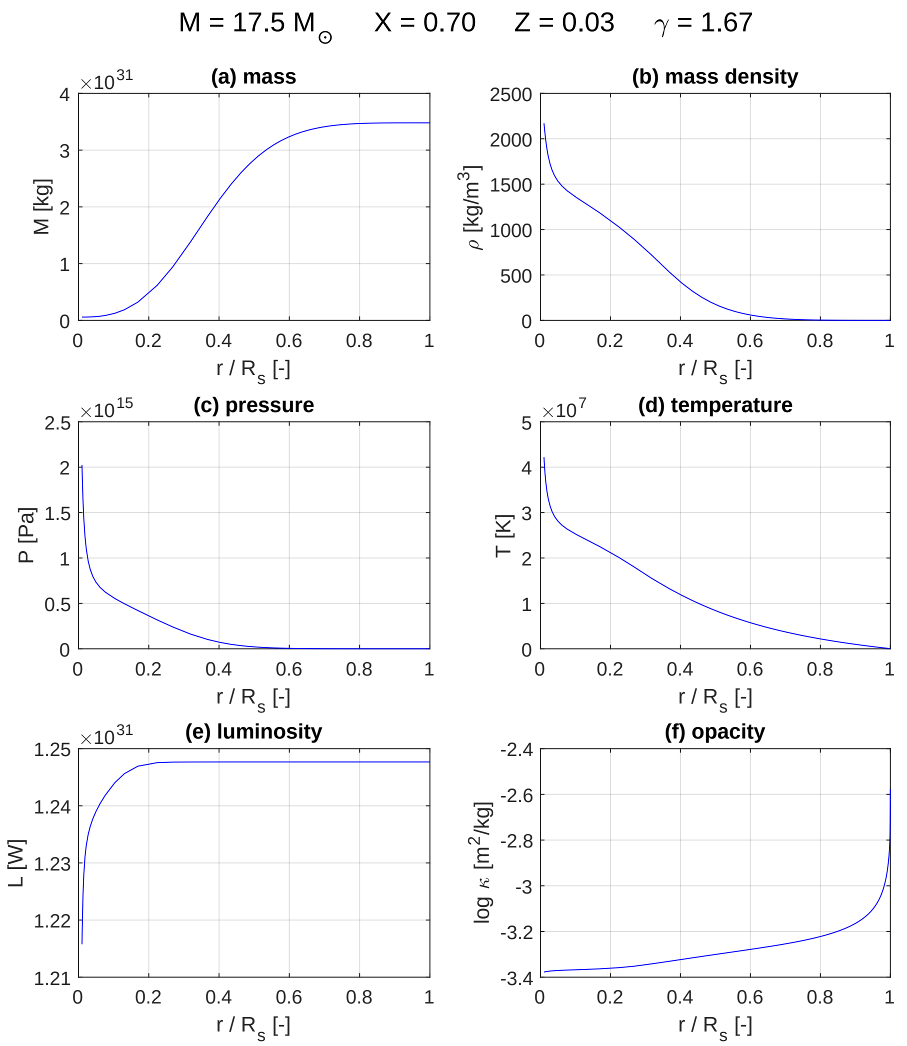

Figure 10: Solution of the simplified model for a $17.5\ \mathrm{M}_\odot$ main sequence star.

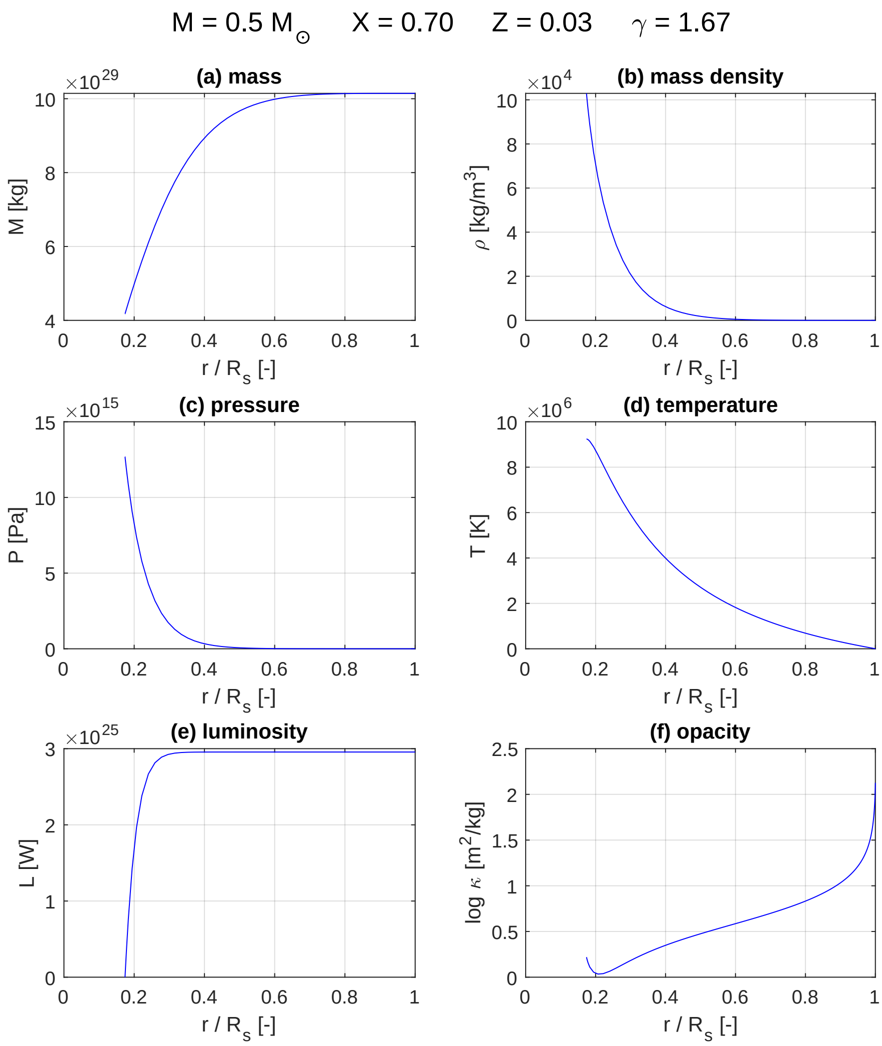

Figure 11: Solution of the simplified model for a $0.51\ \mathrm{M}_{\odot}$ main sequence star.

6 Conclusions

It is possible to build a mathematical model of a zero age main sequence star which fairly matches professional models, even with following simplifications:

- the star is spherically symmetrical; and
- the stellar interior behaves like an ideal gas (allowing the use of the ideal gas law) in which atoms are not bound to each other (allowing the use of $\gamma = 5/3$ for the adiabatic index); and
- the chemical composition is uniform throughout the star; and
- the stellar interior is fully ionized (allowing the use of approximation (13) for the mean molecular weight); and
- convection is considered to be purely adiabatic (allowing the use of the adiabatic gas law); and
- pressure and mass density are assumed to be practically zero at the surface of the star.

Intended further refinements of the subject simplified model include:

- the use of tabulated opacities instead of the current approximations; and/or
- the implementation of a boundary value solver to balance backward and forward solutions, eliminating the need for prior knowledge regarding certain boundary values.

A Equation summary

A.1 Differential equations

A.1.1 Conservation of mass

$$\frac{dM}{dr} = \rho \, 4 \, \pi \, r^2$$

A.1.2 Hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{G \, M \, \rho}{r^2}$$

A.1.3 Conservation of energy

$$\frac{dL}{dr} = \epsilon \, \rho \, 4 \, \pi \, r^2$$

A.1.4 Energy transport

$$\frac{dT}{dr} = -\frac{3 \, \kappa \, \rho \, L}{16 \, \pi \, a \, c \, r^2 \, T^3}$$

$$\frac{dT}{dr} = \left(1 - \frac{1}{\gamma}\right) \frac{T}{P} \, \frac{dP}{dr}$$

A.2 Constitutive relations

A.2.1 Equation of state

$$\mu \approx \frac{1}{2 \, X + \frac{3}{4} \, Y + \frac{1}{2} \, Z}$$

$$\rho = \left(P - \frac{1}{3} \, a \, T^4\right) \frac{\mu \, m_u}{k_B \, T}$$

A.2.2 Energy production rate

$$\epsilon \approx \underbrace{0.241 \, \rho \, X^2 \, T_6^{-2/3} \, e^{-33.80 \, T_6^{-1/3}}}_{\epsilon_{pp}} + \underbrace{8.67 \times 10^{20} \, \rho \, X \, X_{CNO} \, T_6^{-2/3} \, e^{-152.28 \, T_6^{-1/3}}}_{\epsilon_{CNO}}$$

A.2.3 Opacity

$$\kappa \approx \underbrace{0.02 \, (1 + X)}_{\kappa_{es}} + \underbrace{3.68 \times 10^{18} \, (1 - Z) \, (1 + X) \, \rho \, T^{-3.5}}_{\kappa_{ff}} + \underbrace{4.34 \times 10^{21} \, \frac{1}{10} \, Z \, (1 + X) \, \rho \, T^{-3.5}}_{\kappa_{bf}}$$

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